Low-dimensional linear representations of mapping class groups and their triviality in certain cases

Thesis submitted to The University of Bern for the degree of Bachelor of Mathematics

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Dedicated to Professor Dr. Sebastian Baader

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Abstract. This Bachelor-Thesis gives an overview of mapping class groups and linear representations thereof. The main goal of the thesis is to understand and explain the proof of Mustafa Korkmaz, saying that for $g \geq 3$ and $n \leq 2g-1$, every homomorphism from the mapping class group of an orientable surface of genus g to $\mathrm{GL}(n, \mathbb{C})$ is trivial.[**Kor11**]

Contents

Preface

This Bachelor-Thesis gives an overview of mapping class groups and linear representations thereof. The main goal of the thesis is to understand and explain the proof of Mustafa Korkmaz, saying that for $g \geq 3$ and $n \leq 2g-1$, every homomorphism from the mapping class group of an orientable surface of genus g to $\mathrm{GL}(n, \mathbb{C})$ is trivial.[**Kor11**]

In the first part of the thesis there will be given a general overview and introduction to the mapping class group. The second part will deal with the proof of Korkmaz and is based largely on his recent paper [**Kor11**], explaining and extending parts of it.

I tried to include most of the definitions of the mathematical vocabulary used in this paper, but a knowledge about linear algebra will be essential for the understanding of some of the proofs given.

This thesis was typeset using LaTeX. Citations were managed using Citavi[1], Zotero[2] and BibTeX. Revision control using Mercurial[3], TortoiseHG[4] and bitbucket[5] was used during the process of creation of this thesis.

Graphics were mostly done by hand and and some of them in colour. I apologize at this point that some of those graphics might be less-comprehensible if the thesis is printed or viewed without colour, even though it isn't essential to understand the graphics to comprehend the text. The graphics merely are there as examples and complement the text to illustrate points and help visualizing concepts.

[1] http://www.citavi.com/
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[4] http://tortoisehg.bitbucket.org/
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Part 1

Mapping Class Groups

CHAPTER 1

Introduction and Definitions

Drink wine, this is life eternal,
This, all that youth will give you:
It is the season for wine, roses and friends drinking together.
Be happy for this moment – it is all life is.

– *Omar Khayyám*

1. Surfaces, Curves and Homeomorphisms

DEFINITION 1. *Let $f : X \to Y$ be a function between two topological spaces. If f is continuous, bijective and f^{-1} is continuous, then f is called a* ***homeomorphism***[1] *and the spaces X and Y are called* ***homeomorphic****.[**Mun00**]*

REMARK 1. *A homeomorphism stretches and deforms the space continuously without puncturing or breaking it.*

The set of all self-homeomorphisms of a topological space X, together with the operation of function composition build a group $\mathrm{Homeo}(X)$ called the homeomorphism group of X[**Haz88**].

DEFINITION 2. *A* ***surface*** *S is a 2-dimensional manifold (short 2-manifold). The meaning of this is that S is hausdorff, 2nd-countable and locally homeomorphic to euclidean space.[**FM12**]*

EXAMPLE 1. *Figures 1, 2, 3 and 4 show various examples of surfaces.*

DEFINITION 3. *The product $A := S^1 \times [0, 1]$ is called (closed)* ***annulus****.*

EXAMPLE 2. *Figure 5 shows an example of an annulus.*

REMARK 2. *A homeomorphism between surfaces is orientation preserving if the face of an imagined clock on the surface is only rotated and moved, streched and so on by the homeomorphism but not mirrored.*

[1] I would have preferred the term **bicontinuous function** or **topological isomorphism** to avoid confusion with homomorphism, but the term homeomorphism is widely used through literature.

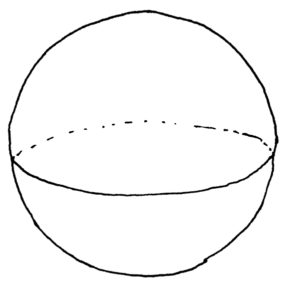

FIGURE 1. A genus g=0 surface

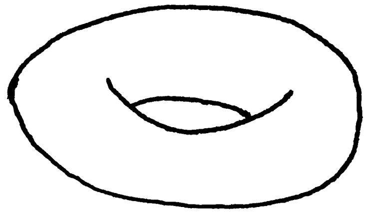

FIGURE 2. A genus g=1 surface

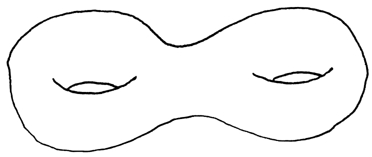

FIGURE 3. A genus g=2 surface

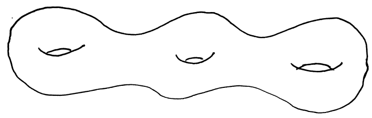

FIGURE 4. A genus g=3 surface

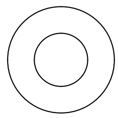

FIGURE 5. An annulus

Geometrically the annulus is a closed disc with a smaller open disc cut out from the center, such that the resulting surface is closed. An annulus can be embedded in a plane, if the plane is parametrised using polar coordinates (φ, r) then $(\varphi, r) \mapsto (\varphi, r+1)$ is an embedding. Topologically the (closed) annulus is equivalent to a (closed) cylinder.

For an oriented surface S let $\mathrm{Homeo}^+(S)$ denote the group of orientation preserving homeomorphisms of S.

By results of Möbius and Radò (for a proof see for example [**Tho92**]) surfaces are classified by the following theorem:

THEOREM 1 (Classification of surfaces). *Let S be a compact, connected, orientable surface. S is classified up to homeomorphism by the number of **boundary components** $b \geq 0$ and the **genus** $g \geq 0$ of the surface. Starting from a 2-dimensional sphere, the genus g specifies how many tori have been connected to it, while the number b specifies how many open discs with disjoint closures have been removed from it. [**FM12, Tho92**]*

A surface may also be punctured by removing n points from the interior or equivalently (up to homeomorphism) by removing n closed disjoint discs.

DEFINITION 4. *A continuous map $\alpha : S^1 \to S$ to a surface is called a* ***closed curve*** *on S. If α is injective (e.g. the curve doesn't have any self-intersections) then it is called* ***simple****. In the text a closed curve will usually be identified with its image in S.[**FM12**]*

DEFINITION 5 (Homotopy and isotopy of functions). *Given two continuous functions $f : X \to Y$ and $g : X \to Y$ between two topological spaces, if there is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, then H is called homotopy and f and g are called homotopic. If X and Y are topological spaces and for each $t \in [0, 1]$ $H|_{X\times t} =: H_t : X \to Y$ is a homeomorphism then H is called isotopy, and f and g are called isotopic.[**FM12, Mun00**]*

REMARK 3 (Homotopy and isotopy of curves). *Especially for simple closed curves we have that: Let α and β be two simple closed curves. If there exists a continuous function $H : S^1 \times [0, 1] \to S$, H is called* ***homotopy****, and α and β are called* ***homotopic****. If there is a H such that $\forall t \in [0, 1]$ the closed curve $H(S^1 \times t)$ is simple, then H is called* ***isotopy****, and α and β are called* ***isotopic****.[**FM12, Mun00**]*

DEFINITION 6. *A closed curve α that is not homotopic to a point, puncture or boundary component is called* ***essential****. [**FM12**]*

DEFINITION 7. *Let $Homeo_0(S)$ denote the normal subgroup of $Homeo(S)$ of homeomorphisms isotopic to the identity.*

PROPOSITION 1. *Let α and β be two essential simple closed curves in a surface S, then:*

$$\alpha \text{ isotopic to } \beta \Leftrightarrow \alpha \text{ homotopic to } \beta$$

PROOF. For a proof see for example [**FM12**]. □

Homotopy is clearly an equivalence relation [**Mun00**]. Thus it is possible to define an equivalence class on homotopic curves.

NOTATION 1. *Given a simple closed curve α, let $[\alpha]$ denote its homotopy equivalence class. If it is clear from the context a simple closed curve is usually identified with its homotopic equivalence class leaving out the square brackets.*

REMARK 4. *The definition of a closed curve and its homotopy classes can be generalised as follows: Let $f : S^n \to S$ be a continuous map and let $b_0 \in im(f)$ and $a_0 \in S^n$ with $f(a_0) = b_0$. For $n \geq 1$ the set of homotopy classes of such maps is a group $\pi_n(S, b_0)$ called n-th homotopy group. The group operation is given by joining two curves together at the basepoint. For an exact definition see [**Hat02**].*

DEFINITION 8. *Given two simple closed curves α and β on a surface S, where $[\alpha]$ and $[\beta]$ are the respective homotopy classes. Define the* ***geometric intersection number*** *as*

$$i([\alpha], [\beta]) = \min\{|\alpha \cap \beta| : \alpha \in [\alpha], \beta \in [\beta]\}$$

*by abuse of notation $i(\alpha, \beta)$ will mean $i([\alpha], [\beta])$.[**FM12**]*

Given a simple closed curve in a surface it is intuitively possible to cut the surface open along the curve. The resulting surface can be either connected or not. Given two homotopic simple closed curve, cutting along one curve will always yield a surface which is homeomorphic to the surface resulting by cutting about the other curve.

DEFINITION 9. *Let α be a simple closed curve on a surface S. Let S_α denote the surface obtained by* ***cutting*** *S along α, such that there is an homeomorphism h between the two resulting boundary components and* ***gluing*** *along h – meaning*

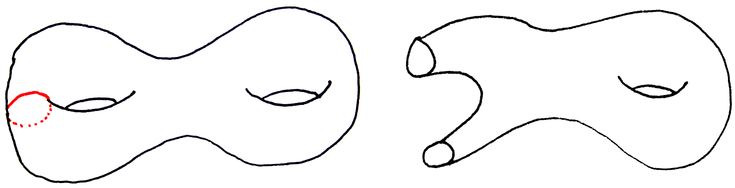

FIGURE 6. A nonseparating curve α in a surface S and the resulting cut surface S_α

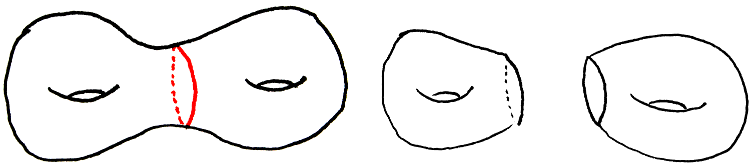

FIGURE 7. A separating curve α in a surface S and the resulting cut (separated) surface S_α

building the quotient $S_\alpha/(x \sim h(x))$ – will yield back a surface homeomorphic to S such that the image of the boundary components under this quotient map is α. If the resulting surface S_α is connected, then α is called ***nonseparating*** *otherwise* ***separating.[FM12]***

EXAMPLE 3. *Figures 6 and 7 show examples of nonseparating respectively separating curves in a surface and their resulting cut surface.*

PROPOSITION 2. *Any two nonseparating simple closed curves α, β in a surface S yield homeomorphic S_α, S_β. And there is a homeomorphism $h : S \to S$ of S such that $h(\alpha) = \beta$.* ***[FM12]***

PROOF. Given any two nonseparating simple closed curves α and β in a surface S with genus g, let S_α and S_β denote the corresponding surfaces resulting by cutting along α respectiviely along β. The resulting surfaces will have the same number of boundary components and the same number of punctures. While the genus of the cut surfaces will be $g-1$. Thus by the classification of surfaces S_α is homeomorphic to S_β. By the definition of the cut surfaces there are homeomorphisms h_α and h_β mapping each of the two cut boundary components in the respective surface. Choose a homeomorphism $h : S_\alpha \to S_\beta$ respecting the equivalences $x \sim h_\alpha(x)$ and $y \sim h_\beta(y)$. Thus this extends to a homeomorphism $S \to S_\alpha \to S_\beta \to S$ on S, taking α to β. □

The same is not true for separating curves.

PROPOSITION 3 (Classification of simple closed curves on a surface[**FM12**]). *Let α and β be two simple closed curves on a surface S and let S_α and S_β denote the corresponding cut surfaces. Then the following two conditions are equivalent:*

(1) *There is an orientation-preserving homeomorphism $h : S \to S$ such that $h(\alpha) = \beta$.*
(2) *There is a homeomorphism $h : S_\alpha \to S_\beta$*

PROOF. See [**FM12**] for a proof. □

COROLLARY 1. *Any nonseparating simple closed curve is essential.*

REMARK 5. *For a family of simple closed curves on S, the intersection pattern of the family roughly means the appearance of the graph induced by the intersections of the curves. Where each intersection is a vertex and the connecting curves between each intersection are the edges. For two families $\mathcal{A}$ and $\mathcal{B}$ of simple closed curves on S to have the same intersection pattern, means that each family can be ordered in an n-tuple $(\alpha_1, \alpha_2, \dots, \alpha_n)$ respectively $(\beta_1, \beta_2, \dots, \beta_n)$ such that for any two indices $i, j \in \{1, \dots, n\}$ the intersection number of the two elements is equal: $i(\alpha_i, \alpha_j) = i(\beta_i, \beta_j)$.*

PROPOSITION 4 (Change of coordinates principle[**FM12**]). *Let $\mathcal{A} = (\alpha_i)_i$ and $\mathcal{B} = (\beta)_i$ be two families of simple closed curves on S. Such that $\mathcal{A}$ and $\mathcal{B}$ have the same intersection pattern ($i(\alpha_i, \alpha_j) = i(\beta_i, \beta_j)$). Then there exists a self-homeomorphism $h : S \to S$ such that $h(\alpha_i) = \beta_i$.*

2. Mapping Class Groups

Given a surface S one is interested to study certain manipulations on it, namely self-homeomorphisms of the surface. The group $\mathrm{Homeo}(S)$ was already defined earlier. Recall that

- $\mathrm{Homeo}(S)$ is the group of self-homeomorphisms of the surface S
- $\mathrm{Homeo}^+(S)$ is the group of orientation-preserving self-homeomorphisms of the surface S
- $\mathrm{Homeo}_0(S)$ is the normal subgroup of $\mathrm{Homeo}(S)$ of homeomorphisms isotopic to the identity.
- Let $\mathrm{Diff}^+(S, \partial S)$ be the group of orientation-preserving diffeomorphisms of S that are the identity on the boundary.
- Let $\mathrm{Homeo}(S, \partial S)$, $\mathrm{Homeo}^+(S, \partial S)$ and $\mathrm{Homeo}_0(S, \partial S)$ denote the respective groups of homeomorphisms but restricting to the identity on ∂S

DEFINITION 10 (Compact-open topology on $\mathrm{Homeo}^+(S)$, see for example [**CR78**] and [**Sch75**]). *Let S be a surface. For a compact subset K of S and an open subset U of S define $\Omega(K, U) := \{f \in \mathrm{Homeo}^+(S) : f(K) \subset U\}$. Then the compact-open topology on $\mathrm{Homeo}^+(S)$ is generated by all sets of the form $\Omega(K, U)$ with $K \subset S$ compact and $U \subset S$ open.*

In the following text it will usually be assumed that $\mathrm{Homeo}^+(S)$ is endowed with the compact-open topology. The group $\mathrm{Homeo}^+(S)$ itself is usually too big to study. Intuitively it makes sense to treat homeomorphisms isotopic to each other the same. For example if for all closed simple essential curves on a surface and a homeomorphism taking those curves to some curves, if we can deform those curves without cutting them and without self intersections to get back the original curves, then the homeomorphism was not so interesting to begin with. We therefore build the quotient $\mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S)$.

DEFINITION 11 (Mapping Class Group, [**FM12**]). *Let S be a surface. Then let*

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S)$$

this group is called the ***mapping class group*** *of S. Elements of $\mathrm{Mod}(S)$ are called* ***mapping classes****.*

NOTATION 2. *Elements of the mapping class group are applied right to left in accordance to notation for functions.*

The mapping class group was first studied by Max Dehn[**Deh38, Deh87**].

PROPOSITION 5. *The following definitions are equivalent (up to isomorphism):*

(1) $\mathrm{Mod}(S) := \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S)$

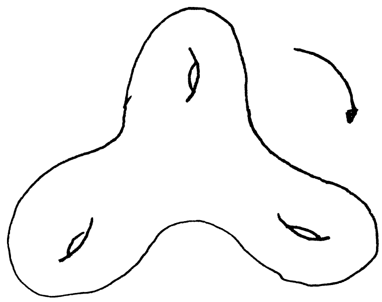

FIGURE 8. An order 3 element of the mapping class group of a genus $g = 3$ surface

(2) $\mathrm{Mod}(S) := \pi_0(\mathrm{Homeo}^+(S, \partial S))$
(3) $\mathrm{Mod}(S) := \pi_0(\mathrm{Diff}^+(S, \partial S))$

REMARK 6. *There are different definitions of the mapping class used throughout literature. For example one can either study oriented or unoriented manifolds. The mapping class group of the oriented manifolds is an index 2 subgroup of the mapping class group of unoriented manifolds.*

DEFINITION 12. *Let S be a surface with n punctures (marked points) in the interior. Then denote by $\mathrm{PMod}(S)$ the subgroup of $\mathrm{Mod}(S)$ consisting of elements that fix each puncture individually. This group is called the **pure mapping class group**.*

REMARK 7. *If S doesn't have any punctures then $\mathrm{PMod}(S) = \mathrm{Mod}(S)$.*

PROPOSITION 6 ([**FM12**]). *There is an isomorphism between the mapping classes of following surfaces and groups as follows (For a proof see [**FM12**]):*

- *The mapping class group of the annulus is isomorphic to $\mathbb{Z}$*
- *The mapping class group of the torus is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$*

EXAMPLE 4. *The mapping class group of the sphere is trivial $\mathrm{Mod}(S^2) = 1$.[**Bir74, FM12**]*

EXAMPLE 5. *Given a genus $g = 3$ surface, figure 8 shows an order 3 element of $\mathrm{Mod}(S)$. Given the fundamental polygon of the torus, a rotation of the polygon by $\pi/2$ gives an element of order 4 in the mapping class group of the torus. If the mapping class group of the torus is identified with $\mathrm{SL}(2, \mathbb{Z})$, then this mapping class would be represented by the element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This is illustrated in fig. 9.*

3. Dehn Twists

As with every group, one is interested to know what the generators of the mapping class group are.

DEFINITION 13 (Twist Map of the Annulus). *Let $T : A \to A$ be the map of the annulus given by $(\varphi, r) \mapsto (\varphi + 2\pi r, r)$. This map is called **(left) twist map** of the annulus. The map $T^{-1} : A \to A, (\varphi, r) \mapsto (\varphi - 2\pi r, r)$ is called **right twist map** of the annulus.[**DS02, FM12, Bir74**]*

EXAMPLE 6. *Figure 10 shows an example of a left twist map of the annulus. Figure 12 shows the same twist map applied to the topologically equivalent cylinder. Figure 11 shows a right twist map on an annulus.*

Now assume assume this twist is done on a cylinder embedded in an arbitrary surface. This brings us to the following definition:

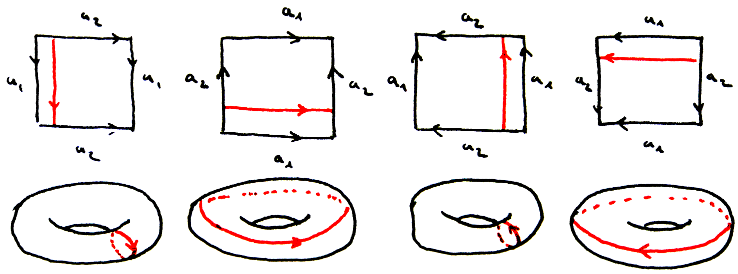

FIGURE 9. An order 4 element of the mapping class group of the torus. The torus is given as a fundamental polygon and embedded in $\mathbb{R}^3$. For visualisation purposes there is an oriented closed curve drawn in red on the torus.

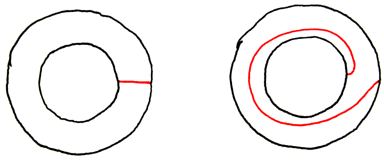

FIGURE 10. A (left) twist map applied to the annulus. The action of the map can be seen in the change of the red curve.

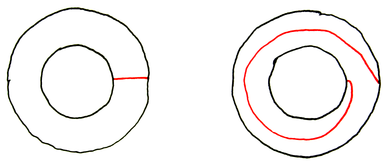

FIGURE 11. A right twist map applied to the annulus. The action of the map can be seen in the change of the red curve.

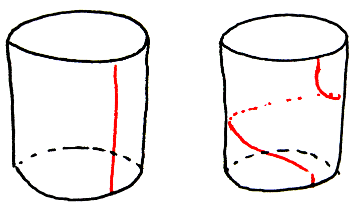

FIGURE 12. A (left) twist map applied to the cylinder which is topologically equivalent to the annulus. The action of the map can be seen in the change of the red curve.

DEFINITION 14 (Dehn Twist, [**FM12**, **CB88**, **DS02**, **Bir74**, **Pap07**]). *Let S be an oriented surface and let α be a simple closed curve in S. Let N be a regular neighbourhood of α and let $\phi : A \to N$ be an orientation preserving homeomorphism. Then define a **dehn twist about** α as the map[2]:*

$$t_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{, if } x \in N \\ x & \text{, if } x \in S - N \end{cases}$$

[2]In accordance with the paper of Korkmaz ([**Kor11**]) Dehn Twists are written using small letters t_α instead of big letters T_α as in the Primer On Mapping Class Groups ([**FM12**])

If this twist is taken modulo isotopy, then it is a well defined element of the mapping class group: $[t_\alpha] = t_{[\alpha]}$.

PROPOSITION 7 (Well-definition of Dehn Twists). *Let* $\alpha, \alpha' \in [\alpha]$ *be two representatives of the isotopy class of simple closed curves in a surface* S. *Let* N *and* N' *be regular neighbourhoods of* α *an* α' *and* $\phi : A \to N$ *and* $\phi' : A \to N'$ *orientation preserving homeomorphisms. Clearly a homotopy* $h : S^1 \times [0,1] \to S$ *between* α *an* α' *can be extended to a homotopy between* ϕ *and* ϕ'.***[FM12]***

NOTATION 3. *In the text* t_α *will usually mean* $t_{[\alpha]}$.

In the text dehn twists will be assumed to be left twists. A dehn twist about a curve α can be understood easily using various equivalent geometrical interpretations. In the first interpretation the curve α of the dehn twist can be seen as an instruction for any curve intersecting it, to first turn left then circle all around α and finally turn right. Another view of the twist operation is, to think of cutting the surface about α then while keeping one neighbourhood of the resulting boundary components fixed, twist the other around by 2π and finally glue them back together. It is important to only twist the neighbourhood of one of the resulting boundary components. If the whole surface would be twisted this would only result in the identity homeomorphism if α is a separating curve.

It turns out that the mapping class group is generated by dehn twists about nonseparating simple closed curves on S. This result was already found by Dehn [**Deh87**] and later improved by W. B. R. Lickorish [**Lic64**].

THEOREM 2 (Dehn-Lickorish Theorem). *Let* S *be a surface of genus* $g \geq 0$. *Then: Dehn twists about finitely many nonseparating simple closed curves generate the mapping class group of* S.

PROOF. For a proof see [**FM12**] or [**Lic64**] or [**Hum79**]. □

PROPOSITION 8. *Let* $[\alpha]$ *and* $[\beta]$ *be two isotopy classes of closed essential simple curves in a surface* S *and let* $n \in \mathbb{Z}$. *Then:*

$$i(t^n_{[\alpha]}([\beta]), [\beta]) = |n| i([\alpha], [\beta])^2$$

PROOF. See [**FM12**] for a proof of the proposition. □

PROPOSITION 9. *Let* $[\alpha]$ *and* $[\beta]$ *be two isotopy classes of simple closed curves. Then:*

$$t_{[\alpha]} = t_{[\beta]} \Leftrightarrow [\alpha] = [\beta]$$

PROOF. "$\Leftarrow$": clear. "$\Rightarrow$": Proof by contraposition, Assume that $a := [\alpha] \neq b := [\beta]$. Then try to find an isotopy class of simple closed curves $c := [\gamma]$ such that $i(a, c) = 0$ and $i(b, c) \neq 0$. Then there are two cases to check:

Case 1 ($i(a, b) \neq 0$)**:** In this case a satisfies the required properties because $i(a, a) = 0$ and $i(b, a) \neq 0$, so with proposition 8 we have $i(t_a(a), a) = i(a, a)^2 = 0$ but $i(t_b(a), a) = i(b, a)^2 \neq 0$, from this it follows that $t_a \neq t_b$ a contradiction.

Case 2 ($i(a, b) = 0$)**:** Here the curve c is neither a nor b, but can be found easily. There are various cases to check. If a is separating, then cut along $\alpha \in a$ to get two surface parts one of genus g' and the other of genus $g - g'$ each with a boundary component, continue on that part of the surface containing b. If b is nonseparating in this cut surface cut along it, then connect the resulting boundary components with an arc γ, re-glue the result will be a curve γ intersecting b once but not a. If b was separating in the cut surface then the genus of the surface containing b must be at least 2 (otherwise $a = b$), because a genus 1 surface doesn't contain a

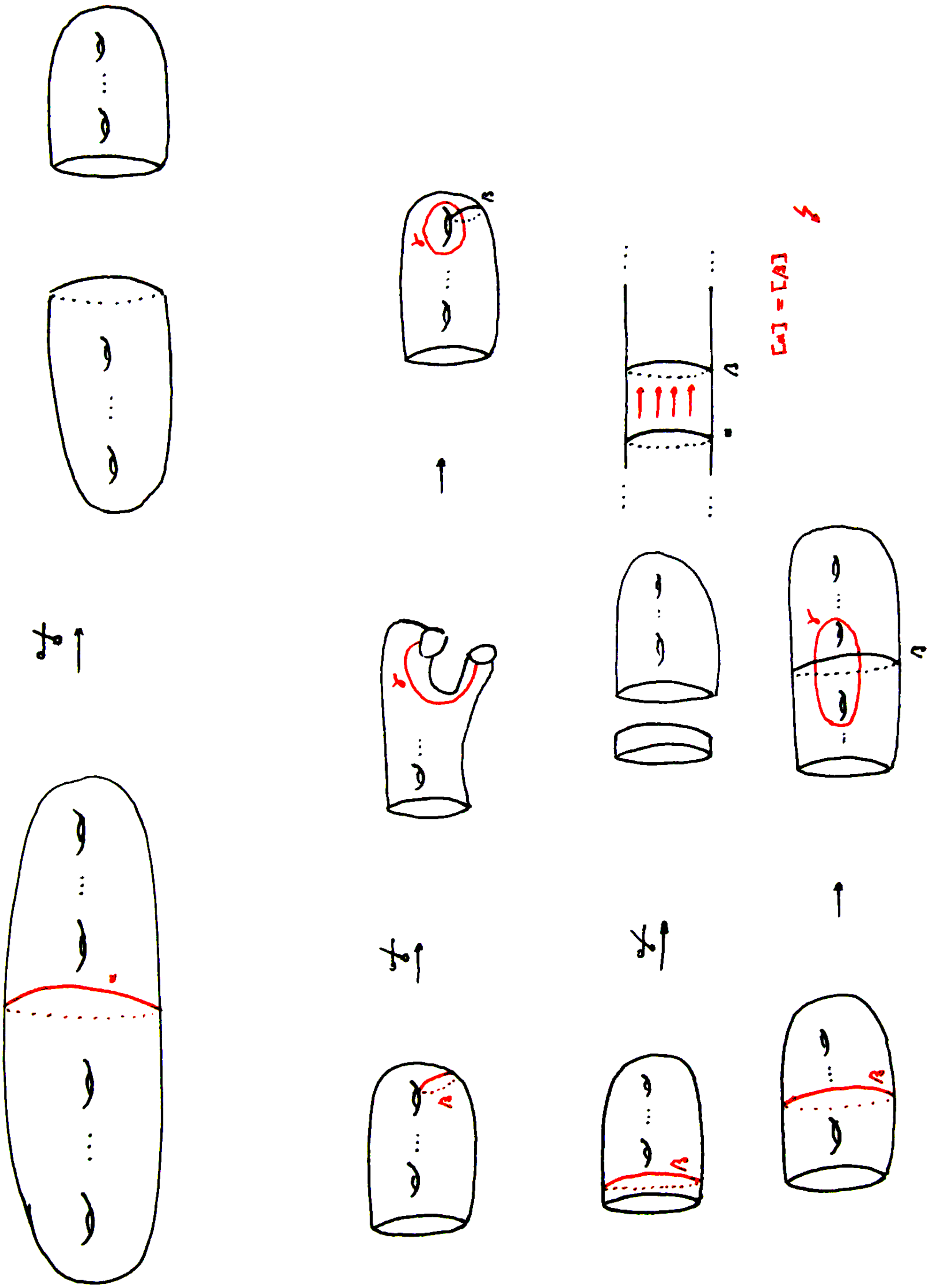

FIGURE 13. The various possibilities are shown if a is separating.

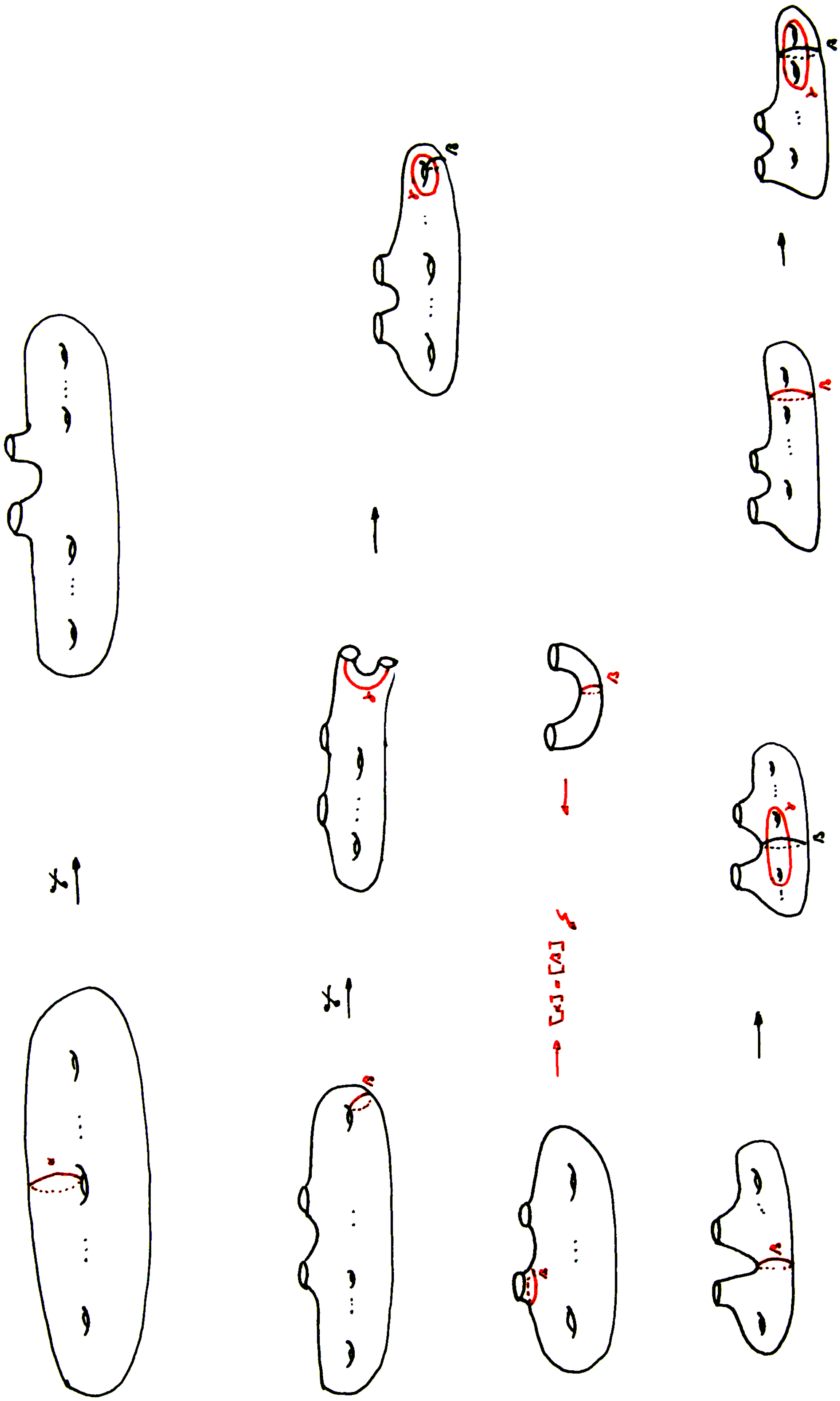

FIGURE 14. The various possibilities are shown if a is nonseparating.

separating curve then there must exist a curve intersecting b. If on the other hand a is nonseparating cutting along it will result in a surface with genus $g-1$ with two boundary components. If b is nonseparating continue like before by cutting along it, connecting the boundary components with an arc and then re-glue everything. If b is separating in the cut surface and if $g-1=1$ then $a=b$ a contradiction. If $g-1>1$ proceed as in the case before. See Figures 13 and 14. Because of the change of coordinates principle, those cases cover all possibilities. Then again similar as in the first case: $i(t_a(c),c)=i(a,c)^2=0$ but $i(t_b(c),c)=i(b,c)^2\neq 0$.

□

LEMMA 1. *For any mapping class $F\in \mathrm{Mod}(S)$ and any isotopy class $[\alpha]$ of simple closed curves in S:*

$$t_{F([\alpha])}=F\circ t_{[\alpha]}\circ F^{-1}$$

PROOF. Let $f\in F$, $\alpha\in[\alpha]$ and $t_\alpha\in t_{[\alpha]}$ be representatives such that: t_α can be written as

$$t_\alpha(x)=\begin{cases}\phi\circ T\circ\phi^{-1}(x) & \text{, if } x\in N\\ x & \text{, if } x\in S-N_\alpha\end{cases}$$

with N_α a regular neighbourhood of α and $\phi: A\to N_\alpha$ and with $\beta=f(\alpha)$. Then for any regular neighbourhood of β, $f\circ t_{[\alpha]}\circ f^{-1}$ takes the neighbourhood to a neighbourhood of α twists it and takes it back to the β-neighbourhood. The result is a twist around $t_\beta=t_{f(\alpha)}$. For any $x\in S-N_\beta$, $f^{-1}(x)\in S-N_\alpha$ and so we have the identity on both sides.**[FM12]** □

COROLLARY 2. *For any mapping class $F\in\mathrm{Mod}(S)$ and any isotopy class $[\alpha]$ of simple closed curves in S:*

$$F([\alpha])=[\alpha]\Leftrightarrow t_{[\alpha]}\circ F=F\circ t_{[\alpha]}$$

PROOF. Just multiply by F^{-1} from the right side and use the lemma from above. □

DEFINITION 15 (Conjugacy). *If $[\alpha]$ and $[\beta]$ are isotopy classes of nonseparating simple closed curves in S, and there exists a $F\in\mathrm{Mod}(S)$ such that:*

$$t_\beta=F\circ t_\alpha\circ F^{-1}$$

then $[\alpha]$ and $[\beta]$ are called ***conjugate.[FM12]***

COROLLARY 3. *If $[\alpha]$ and $[\beta]$ are isotopy classes of nonseparating simple closed curves in S, then there exists a $F\in\mathrm{Mod}(S)$ such that:*

$$t_\beta=F\circ t_\alpha\circ F^{-1}$$

PROOF. This follows from the fact that nonseparating simple closed curves can be taken to each other by a homeomorphism (Proposition 2), and from the lemma above. That means there exists a homeomorphism F with $F([\alpha])=[\beta]$. □

PROPOSITION 10. *For any two isotopy classes of simple closed curves $[\alpha]$ and $[\beta]$ in a surface S:*

$$t_{[\alpha]}\circ t_{[\beta]}=t_{[\beta]}\circ t_{[\alpha]}\Leftrightarrow i([\alpha],[\beta])=0$$

PROOF. Proposition 8 gives: $i(t_{[\alpha]}([\beta]),[\beta])=i([\alpha],[\beta])^2$.

$$\begin{aligned}t_{[\alpha]}\circ t_{[\beta]}=t_{[\beta]}\circ t_{[\alpha]}&\Leftrightarrow t_{[\alpha]}=t_{[\beta]}\circ t_{[\alpha]}\circ t_{[\beta]}^{-1}\\ \Leftrightarrow t_{[\beta]}([\alpha])=[\alpha]&\Rightarrow i(t_{[\beta]}([\alpha]),[\alpha])=i([\alpha],[\alpha])=0\\ \Leftrightarrow 0=i(t_{[\beta]}([\alpha]),[\alpha])&=i([\alpha],[\beta])^2\Rightarrow i([\alpha],[\beta])=0\end{aligned}$$

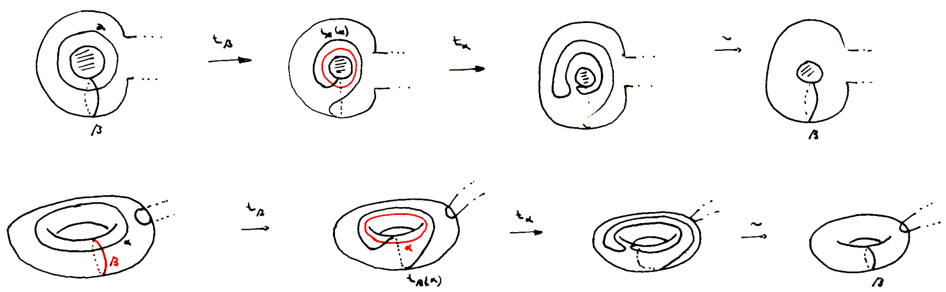

FIGURE 15. Proof of the Braid Relation

For the other direction it is clear that from $i([\alpha],[\beta]) = 0$ it follows that $t_{[\beta]}([\alpha]) = [\alpha]$. □

PROPOSITION 11 (Braid relation, [**FM12, Bir74**]). *If $[\alpha]$ and $[\beta]$ are isotopy classes of simple closed curves in S such that $i([\alpha],[\beta]) = 1$ then:*

$$t_{[\alpha]} \circ t_{[\beta]} \circ t_{[\alpha]} = t_{[\beta]} \circ t_{[\alpha]} \circ t_{[\beta]} \qquad \text{(Braid relation)}$$

PROOF. Multiplying from the right by $t_{[\beta]}^{-1} \circ t_{[\alpha]}^{-1}$ gives:

$$t_{[\alpha]} \circ t_{[\beta]} \circ t_{[\alpha]} \circ t_{[\beta]}^{-1} \circ t_{[\alpha]}^{-1} = t_{[\beta]}$$
$$\Leftrightarrow t_{[\alpha]} \circ t_{[\beta]} \circ t_{[\alpha]} \circ (t_{[\alpha]} \circ t_{[\beta]})^{-1} = t_{[\beta]}$$
$$\Leftrightarrow t_{(t_{[\alpha]} \circ t_{[\beta]})([\alpha])} = t_{[\beta]}$$
$$\Leftrightarrow (t_{[\alpha]} \circ t_{[\beta]})([\alpha]) = [\beta]$$

It suffices to show this for one pair of curves with $i([\alpha],[\beta]) = 1$ because of the change of coordinates principle it then follows for all isotopy classes of curves with $i([\alpha],[\beta]) = 1$. The proof is done in Figure 15. The drawn torus is to be thought of as eventually connected to a bigger surface as indicated by the dotted lines. There is a view given in 3 dimensions and in 2-dimensional view from above. □

Part 2

Representations of Mapping Class Groups

CHAPTER 2

Linear Representations of Mapping Class Groups

> Somebody said
> that Reason was dead.
> Reason said: No,
> I think not so.
>
> – *Piet Hein*

The main work of this chapter will be the proof of the following theorem:

THEOREM 3 (Main Theorem (Korkmaz 2011 [**Kor11**])). *Let S be a compact connected oriented surface of genus $g \geq 1$ with $q \geq 0$ boundary components and let $n \leq 2g-1$. Let $\phi : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ be a homomorphism. Then*

(1) $\mathrm{Im}(\phi) \cong \{I\}$ *if* $g \geq 3$,
(2) $\mathrm{Im}(\phi) \cong \mathbb{Z}_{10}/N$ *where N is a (normal) subgroup of* $\mathbb{Z}_{10}$ *if* $g = 2$,
(3) $\mathrm{Im}(\phi) \cong \mathbb{Z}_{12}/N$ *where N is a (normal) subgroup of* $\mathbb{Z}_{12}$ *if* $g = 1$ *and* $q = 0$,
(4) $\mathrm{Im}(\phi) \cong \mathbb{Z}^q/N$ *where N is a (normal) subgroup of* $\mathbb{Z}^q$ *if* $g = 1$ *and* $q \geq 1$,

1. Linear Algebra and Algebra Preliminaries

1.1. Linear Algebra.

LEMMA 2 ([**Kor11**], Lemma 2.1). *Let* $C = \begin{pmatrix} \lambda & * & * \\ 0 & \lambda & * \\ 0 & 0 & \lambda \end{pmatrix}$ *and* $D = \begin{pmatrix} \mu & * & * \\ 0 & \mu & * \\ 0 & 0 & \mu \end{pmatrix}$ *be two elements of* $\mathrm{GL}(3, \mathbb{C})$. *Then*

$$CDC = DCD \Leftrightarrow C = D$$

PROOF. "$\Leftarrow$": clear.
"$\Rightarrow$": Let $C = \begin{pmatrix} \lambda & a & b \\ 0 & \lambda & c \\ 0 & 0 & \lambda \end{pmatrix}$ and $D = \begin{pmatrix} \mu & d & e \\ 0 & \mu & f \\ 0 & 0 & \mu \end{pmatrix}$ Then

$$CDC = \begin{pmatrix} \lambda^2\mu & a\lambda\mu + \lambda(d\lambda + a\mu) & b\lambda\mu + c(d\lambda + a\mu) + \lambda(af + e\lambda + b\mu) \\ 0 & \lambda^2\mu & c\lambda\mu + \lambda(f\lambda + c\mu) \\ 0 & 0 & \lambda^2\mu \end{pmatrix}$$

and

$$DCD = \begin{pmatrix} \lambda\mu^2 & d\lambda\mu + (d\lambda + a\mu)\mu & e\lambda\mu + (cd + e\lambda + b\mu)\mu + f(d\lambda + a\mu) \\ 0 & \lambda\mu^2 & f\lambda\mu + (f\lambda + c\mu)\mu \\ 0 & 0 & \lambda\mu^2 \end{pmatrix}$$

Comparing the top left entries gives (since $\lambda\mu \neq 0$)

$$\lambda^2\mu = \lambda\mu^2 \Rightarrow \underline{\lambda = \mu}$$

continuing and using this result yields

$$a\lambda\mu + \lambda(d\lambda + a\mu) = d\lambda\mu + (d\lambda + a\mu)\mu$$
$$\Leftrightarrow a\lambda^2 + \lambda(d\lambda + a\lambda) = d\lambda^2 + (d\lambda + a\lambda)\lambda$$

$$\Leftrightarrow a\lambda^2 = d\lambda^2$$
$$\Leftrightarrow \underline{a = d}$$

and

$$c\lambda\mu + \lambda(f\lambda + c\mu) = f\lambda\mu + (f\lambda + c\mu)\mu$$
$$\Leftrightarrow c\lambda^2 + \lambda(f\lambda + c\lambda) = f\lambda^2 + (f\lambda + c\lambda)\lambda$$
$$\Leftrightarrow \underline{c = f}$$

and

$$b\lambda\mu + c(d\lambda + a\mu) + \lambda(af + e\lambda + b\mu) = e\lambda\mu + (cd + e\lambda + b\mu)\mu + f(d\lambda + a\mu)$$
$$\Leftrightarrow b\lambda^2 + c(2a\lambda) + \lambda(ac + e\lambda + b\lambda) = e\lambda^2 + (ca + e\lambda + b\lambda)\lambda + c(2a\lambda)$$
$$\Leftrightarrow \underline{b = e}$$

□

LEMMA 3. *Let Let* $C = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & u \\ 0 & 0 & \mu \end{pmatrix}$ *and* $D = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & v \\ 0 & 0 & \mu \end{pmatrix}$ *be two elements of* $\mathrm{GL}(3, \mathbb{C})$. *Then* $CD = DC$.

PROOF. A calculation shows:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & u \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & v \\ 0 & 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \mu^2 & u\mu + v\mu \\ 0 & 0 & \mu^2 \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & v \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & u \\ 0 & 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \mu^2 & u\mu + v\mu \\ 0 & 0 & \mu^2 \end{pmatrix}$$

□

REMARK 8. *A scalar matrix* λI *commutes with every other matrix. Any two matrices with entries only on the main diagonal commute with each other.*

LEMMA 4. *Let* C *and* D *be two similar matrices in* $\mathrm{GL}(n, \mathbb{C})$. *Meaning, there is an invertible matrix* F *in* $\mathrm{GL}(n, \mathbb{C})$ *such that* $D = F^{-1}CF$. *Then* C *and* D *have the same:*

- *Eigenvalues*
- *Determinant*
- *Trace*
- *Rank*
- *Characteristic polynomial*

Furthermore, for every matrix E *in there is a matrix* J *similar to* E *which is in Jordan form.*

PROOF. The proof is only done for the characteristic polynomial, eigenvalues and determinant. Let $p_C(x)$ and $p_D(x)$ be the respective characteristic polynomials. Then $p_C(x) = \det(xI - C)$, because of similarity there exists a F such that $D = F^{-1}CF$, therefore $\det(xI - C) = \det(xI - FDF^{-1})$. Doing the trick $I = FF^{-1}$ on the identity matrix gives $\det(xI - FDF^{-1}) = \det(FxIF^{-1} - FDF^{-1}) = \det(F(xI - D)F^{-1}) = \det(F)\det(xI - D)\det(F^{-1}) = \det(xI - D)$.[**HJ85**] It follows directly that C and D have the same eigenvalues. Because the determinant is a multiplicative map it is straightforward that both matrices share the same determinant. For the proof of the existence of the jordan form over complex numbers see for example [**HJ85**] or [**Koe97**]. □

REMARK 9. *Note that any two Dehn twists about isotopy classes of nonseparating simple closed curves are conjugate and therefore their representing matrices similar.*

DEFINITION 16. *Given a linear map $L : V \to V$ of a vector space V and a subspace $V' \subseteq V$. If $L(V') \subseteq V'$ then V' is called L-**invariant**.*

1.2. Algebra.

DEFINITION 17. *A group G is called **solvable** iff it has a finite subnormal series whose factor groups are all abelian. Concretely it means that there are subgroups $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{m-1} \lhd G_m = G$ such that $\forall i \in \{1, \ldots, m\} : G_i/G_{i-1}$ is abelian. Here $G_{i-1} \lhd G_i$ means G_{i-1} is normal in G_i. [**Rot94**]*

DEFINITION 18. *For a group G let $[G, G]$ denote the **commutator subgroup** of G defined by: $[G, G] = \big\langle \{xyx^{-1}y^{-1} : x, y \in G\} \big\rangle$. [**Lan02**]*

LEMMA 5 ([**Kor11**], Lemma 2.2). *The subgroup $\Delta\mathrm{GL}(n, \mathbb{C})$ of $\mathrm{GL}(n, \mathbb{C})$ consisting of upper triangular matrices is solvable.*

PROOF. Let $G_0 := \Delta\mathrm{GL}(n, \mathbb{C})$ and $G_i := [G_{i-1}, G_{i-1}]$. Because the commutator subgroups are normal it is sufficient to show that there is a $G_j = I$, the resulting quotients are obviously abelian (abelianised) by factoring with the commutator subgroup. Let $C = (c_{ij})$ be an element of $\Delta\mathrm{GL}(n, \mathbb{C})$. Because $\det(C) \neq 0$, C does not have any 0 entries on the main diagonal. Let $E := C^{-1}$ then for the main diagonal entries it follows that: $e_{ii} = c_{ii}^{-1}$. So for any two elements C and D of $\Delta\mathrm{GL}(n, \mathbb{C})$, the main diagonal entries of $CDC^{-1}D^{-1}$ must be all 1. So any element of G_1 has the form

$$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

For any two elements C and D of G_1 we find

$$\begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & & & \vdots \\ 0 & 0 & \ddots & a_{i-1,i} & * \\ \vdots & & \ddots & 1 & * \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & & & \vdots \\ 0 & 0 & \ddots & b_{i-1,i} & * \\ \vdots & & \ddots & 1 & * \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & & & \vdots \\ 0 & 0 & \ddots & a_{i-1,i} + b_{i-1,i} & * \\ \vdots & & \ddots & 1 & * \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

for the entries above the main diagonal. So $(CDC^{-1}D^{-1})_{i-1,i} = c_{i-1,i} + d_{i-1,i} - c_{i-1,i} - d_{i-1,i}$. Therefore every element from G_2 must look like:

$$\begin{pmatrix} 1 & 0 & * & \cdots & * \\ 0 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & * \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Repeating this procedure will move zeros up by one diagonal in each step. So $G_n = \{I\}$. $\square$

DEFINITION 19. *A group G is called **perfect** if it is equal to its commutator subgroup ($G = [G, G]$).[**Ros94**]*

LEMMA 6 ([**Kor11**], Lemma 2.3). *Any homomorphism from a perfect group G to an abelian group H is trivial.*

PROOF. Let $\phi : G \to H$ be any homomorphism from a perfect group to an abelian group. Because H is abelian we have for any two elements g_1 and g_2 of G that

$$\phi(g_1)\phi(g_2) = \phi(g_2)\phi(g_1) \Rightarrow \phi(g_1g_2) = \phi(g_2g_1) \Rightarrow \phi(g_1g_2)\phi(g_2g_1)^{-1} = I$$
$$\Rightarrow \phi(g_1g_2g_1^{-1}g_2^{-1}) = I$$

But every element of G is a product of commutators. □

LEMMA 7 ([**FH11**], Lemma 2.2). *If G is a perfect group and H is a solvable group. Then any homomorphism $G \to H$ is trivial.*

PROOF. Let $\phi : G \to H$ be any homomorphism. Because H is solvable we have a subnormal series with abelian factor groups: $\{I\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_m = H$. Let $f : H \to H/H_{m-1}$ be the canonical map. H/H_{m-1} is abelian, so $f \circ \phi$ is trivial. Therefore $\phi(G) \subseteq H_{m-1}$. Continuing with $H_{m-1} \to H_{m-1}/H_{m-2}$ we get $\phi(G) \subseteq H_{m-2}$. By induction and because we know that $H_0 = \{I\}$ we get $\phi(G) \subseteq H_0 = \{I\}$. □

DEFINITION 20. *For a group G and a vector space V over a field K, with $n = dim(V)$. A homomorphism $\phi : G \to \mathrm{GL}(V)$ is called a* ***representation of** G* ***on** V.[JL93][GW99]*

2. Mapping Class Group Preliminaries

REMARK 10. *Note that in this chapter, S will mean an oriented surface with $p \geq 0$ punctures (marked points) and $q \geq 0$ boundary components, unless otherwise indicated. Furthermore $\mathrm{Mod}(S)$ is supposed to mean $\mathrm{PMod}(S)$ if the surface has marked points.*

NOTATION 4. *For any two F and G in $\mathrm{Mod}(S)$, $G \circ F$ will be written as GF. For any isotopy class of closed simple curves $[\alpha]$, latin letters will denote the class $a = [\alpha]$ and greek letters will denote a representing curve $\alpha \in a$. If there is no danger of confusion a might sometimes be used to denote either a representing curve or the isotopy class depending on the context.*

THEOREM 4 ([**KM00**], Theorem 1.2). *Let S be a surface of genus $g \geq 1$ and α and β two nonseparating simple closed curves on S. Then there is a sequence $\alpha = \alpha_0, \ldots, \alpha_k = \beta$ of nonseparating simple closed curves in S such that:*

$$\forall i \in \{0, \ldots, k-1\} : i(\alpha_i, \alpha_{i+1}) = 1$$

THEOREM 5 ([**KM00**], Theorem 2.7). *Let S be a surface of genus $g \geq 2$ and α and β be two nonseparating simple closed curves in S intersecting at exactly one point. Then the commutator subgroup of $\mathrm{PMod}(S)$ is generated normally by $t_\alpha t_\beta^{-1}$*

PROOF. For a proof see Theorems 2.6, 2.7 and 2.8 in [**KM00**]. Note that there it says that the commutator subgroup of $\mathrm{PMod}(S)$ is generated by the collection of all such elements (of the form $t_\alpha t_\beta^{-1}$). But now let α' and β' be any other pair of nonseparating simple closed curves on S intersecting once. Then there is an orientation preserving homeomorphism $f : S \longrightarrow S$ with $f(\alpha) = \alpha'$ and $f(\beta) = \beta'$ and so: $ft_\alpha f^{-1} f t_\beta^{-1} f^{-1} = f t_\alpha t_\beta^{-1} f^{-1} = t_{\alpha'} t_{\beta'}^{-1}$. Therefore only one $t_\alpha t_\beta^{-1}$ is sufficient to normally generate the commutator subgroup:

$$\left\langle t_\alpha t_\beta^{-1} \right\rangle^G = \left\langle \{g t_\alpha t_\beta^{-1} g^{-1} : g \in \mathrm{PMod}(S)\} \right\rangle$$

□

COROLLARY 4 ([**Kor11**]). *If* $N \triangleleft \mathrm{PMod}(S)$ *is a normal subgroup of* $\mathrm{PMod}(S)$ *with* $t_a t_b^{-1} \in N$ *then:*

$$[\mathrm{PMod}(S), \mathrm{PMod}(S)] \subseteq N$$

THEOREM 6 ([**Pow78**], Theorem 1). *For a surface* S *of genus* $g \geq 3$*, the mapping class group of* S *is perfect (e.g.* $\mathrm{PMod}(S) = [\mathrm{PMod}(S), \mathrm{PMod}(S)]$*).*

THEOREM 7 ([**KM00**], Theorem 4.2). *For a surface* S *of genus* $g \geq 2$*, the commutator subgroup of* $\mathrm{PMod}(S)$ *is perfect:*

$$[\mathrm{PMod}(S), \mathrm{PMod}(S)] = [[\mathrm{PMod}(S), \mathrm{PMod}(S)], [\mathrm{PMod}(S), \mathrm{PMod}(S)]]$$

.

PROOF OF THEOREM 7. By theorem 5 $G' := [\mathrm{PMod}(S), \mathrm{PMod}(S)]$ is generated normally by any element of the form $t_\alpha t_\beta^{-1}$ where α and β are two nonseparating simple closed curves in S intersecting at exactly one point. Because the genus is at least 2, one can find a third nonseparating simple closed curve γ disjoint from α and β. Choose an element of G' as follows:

$$[f, g] \in G' : [f, g] = fgf^{-1}g^{-1}$$

Because of Corollary 3 any two nonseparating simple closed curves are conjugate. So setting $g = t_\beta$ and $f(\beta) = \gamma$ gives $ft_\beta f^{-1} t_\beta^{-1} = t_\gamma t_\beta^{-1} \in G'$ and setting $g = t_\alpha$ and $f(\alpha) = \gamma$ gives $ft_\alpha f^{-1} t_\alpha^{-1} = t_\gamma t_\alpha^{-1} \in G'$. Building the commutator of those two elements one gets that $[t_\gamma t_\beta^{-1}, t_\gamma t_\alpha^{-1}] \in [G', G']$. $[t_\gamma t_\beta^{-1}, t_\gamma t_\alpha^{-1}] = t_\gamma t_\beta^{-1} t_\gamma t_\alpha^{-1} t_\beta t_\gamma^{-1} t_\alpha t_\gamma^{-1}$ because γ is disjoint from α and β it commutes with both of them (Proposition 10). So we get:

$$t_\gamma t_\beta^{-1} t_\gamma t_\alpha^{-1} t_\beta t_\gamma^{-1} t_\alpha t_\gamma^{-1} = t_\beta^{-1} t_\alpha^{-1} t_\beta t_\alpha$$

Extending from the right side with $1 = t_\beta t_\beta^{-1}$ gives:

$$t_\beta^{-1} t_\alpha^{-1} t_\beta t_\alpha = t_\beta^{-1} t_\alpha^{-1} t_\beta t_\alpha t_\beta t_\beta^{-1}$$

here we can use the braid relation ($t_\beta t_\alpha t_\beta = t_\alpha t_\beta t_\alpha$) between α and β because they intersect exactly once, this gives:

$$t_\beta^{-1} t_\alpha^{-1} t_\beta t_\alpha t_\beta t_\beta^{-1} = t_\beta^{-1} t_\alpha^{-1} t_\alpha t_\beta t_\alpha t_\beta^{-1} = t_\alpha t_\beta^{-1}$$

but because G' is generated normally by those elements it follows that $G' \subseteq [G', G']$ and so G' is perfect. □

THEOREM 8 ([**Kor02**], Theorem 5.1). *Let* S *be a surface of genus* $g \geq 1$ *with* $q \geq 0$ *boundary components and* $G := \mathrm{PMod}(S)$ *and* $G' := G/[G, G]$*. Then:*

(1) $G' \cong \{1\}$*, if* $g \geq 3$*,*
(2) $G' \cong \mathbb{Z}_{10}$*, if* $g = 2$*,*
(3) $G' \cong \mathbb{Z}_{12}$*, if* $g = 1$ *and* $q = 0$*,*
(4) $G' \cong \mathbb{Z}^q$*, if* $g = 1$ *and* $q \geq 1$*,*

LEMMA 8. *Let* S *be a surface,* $G := \mathrm{PMod}(S)$ *and* $G^{ab} := G/[G, G]$*. Let* a *and* b *be isotopy classes of nonseparating simple closed curves in* S*. Let* $\phi : G \to G^{ab}, x \mapsto [x]$ *be the canonical homomorphism. Then:*

$$\phi(t_a) = \phi(t_b)$$

PROOF. Because Dehn twists about nonseparating simple closed curves in S are conjugate it follows:

$$\phi(t_b) = \phi(f \circ t_a \circ f^{-1}) = \phi(t_a)$$

□

LEMMA 9 ([**Kor11**], Lemma 3.7). *Let S be a surface of genus $g \geq 1$ and let a and b be isotopy classes of nonseparating simple closed curves on S. If H is an abelian group and $\phi : \mathrm{PMod}(S) \to H$ a homomorphism, then:*

$$\phi(t_a) = \phi(t_b)$$

PROOF. Because Dehn twists about nonseparating simple closed curves in S are conjugate it follows:

$$\phi(t_b) = \phi(f \circ t_a \circ f^{-1}) = \phi(t_a)$$

□

3. Homomorphisms from the Mapping Class Group to $\mathrm{GL}(n, \mathbb{C})$

3.1. Preparations for the proof of the Main Theorem. This section is devoted to the proof of the Main Theorem (Theorem 3). Throughout the following text S is an oriented surface of genus $g \geq 1$ and $n \leq 2g - 1$ and $\phi : \mathrm{PMod}(S) \to \mathrm{GL}(n, \mathbb{C})$ is a homomorphism. This proof is taken from the Paper of Korkmaz[**Kor11**] who based his proof on a proof by Franks and Handel[**FH11**]. Most proofs closely follow the text of Korkmaz[**Kor11**] sometimes extending and explaining it if considered necessary.

NOTATION 5. *In the following text $V := \mathbb{C}^n$ will be a vector space.*

DEFINITION 21. *Let a be an isotopy class of simple closed curves on S. Then the **representative** of the dehn twist about a in $\mathrm{GL}(n, \mathbb{C})$ is denoted by $L_a := \phi(t_a)$. If λ is an eigenvalue of a linear operator L the corresponding eigenspace is denoted by $E_\lambda(L) := \{\vec{v} \in V : L\vec{v} = \lambda\vec{v}\}$. And $E^a_\lambda := E_\lambda(L_a)$.*

LEMMA 10. *Given two linear operations L_a and L_b, if they commute then they preserve each others eigenspaces. In particular if a and b are two nonseparating simple closed curves disjoint from each other, then their linear representatives commute and preserve each others eigenspaces.*

PROOF. Let λ be an eigenvalue of L_a. Assume E^a_λ is not L_b invariant, then $\exists \vec{x} \in E^a_\lambda : (L_b\vec{x}) \notin E^a_\lambda$, so:

$$(I\lambda)(L_b\vec{x}) \neq L_aL_b\vec{x} = L_bL_a\vec{x} = L_b(I\lambda)\vec{x} = (I\lambda)L_b\vec{x}$$

a contradiction. □

REMARK 11. *Generalised eigenspaces ${}^nE^a_\lambda := \mathrm{Kern}((L_a - \lambda I)^n)$ of a linear operator L_a are preserved by a commuting operator. Proof: $(L_a - \lambda I)^n L_b = L_b(L_a - \lambda I)^n$.*

PROPOSITION 12 ([**Kor11**], Proposition 4.1). *Let $g = 2$ and $n \leq 2$. Then $\phi(\mathrm{PMod}(S)) = im(\phi)$ is a quotient of the cyclic group $\mathbb{Z}_{10}$.*

PROOF. $n = 1$**:** Then $\mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$. This group is abelian. Then by the universal property of the abelianisation (See Proof of Lemma 9 for an example), ϕ can be written as: $\phi : \mathrm{PMod}(S) \to \mathbb{C}^* = \phi : \mathrm{PMod}(S) \xrightarrow{\varphi} \mathrm{PMod}(S)/[\mathrm{PMod}(S), \mathrm{PMod}(S)] \to \mathbb{C}^*$. By Theorem 8:

$$\mathrm{PMod}(S)/[\mathrm{PMod}(S), \mathrm{PMod}(S)] = \mathbb{Z}_{10}$$

So $\phi : \mathrm{PMod}(S) \xrightarrow{\varphi} \mathbb{Z}_{10} \to \mathbb{C}^*$. By the Isomorphism Theorem:

$$\mathrm{Im}(\phi) \cong \mathrm{PMod}(S)/\mathrm{Kern}(\phi) \cong \mathbb{Z}_{10}/(\mathrm{Kern}(\phi)/[\mathrm{PMod}(S), \mathrm{PMod}(S)])$$

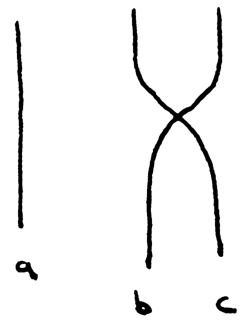

FIGURE 1. Intersection of curves for Proposition 12

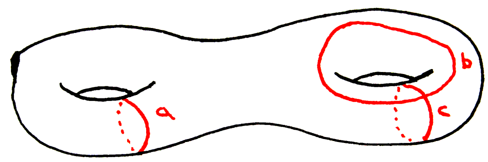

FIGURE 2. Example of how the intersection of curves for Proposition 12 could look like in a genus 2 surface.

$n = 2$: Let $G := \mathrm{PMod}(S)$ and $G' := [\mathrm{PMod}(S), \mathrm{PMod}(S)]$ and $G^{ab} = G/G'$. Let a, b and c be isotopy classes of nonseparating simple closed curves in S, such that a is disjoint from $b \cup c$ and such that b intersects c transversely at one point. Because $g = 2$, those curves clearly exist, that is $i(b,c) = 1$, $i(a,b) = 0$ and $i(a,c) = 0$ as shown in figure 1 and 2. It suffices to proof that $\phi(G') = \{I\}$, because then $G' \subseteq \mathrm{Kern}(\phi)$ and $\mathrm{Im}(\phi) \cong G/\mathrm{Kern}(\phi) \cong G^{ab}/(\mathrm{Kern}(\phi)/G')$. Let L_a be in jordan form, then there are three possibilities (with respect to an adequate basis) for L_a: $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$. Those cases are analysed separately:

(1) If L_a has two distinct eigenvalues $\lambda \neq \mu$ then choose as a basis $(\vec{v_1}, \vec{v_2}) : \vec{v_1} \in E^a_\lambda, \vec{v_2} \in E^a_\mu$. L_a is diagonal, because neither b nor c intersects a, L_b and L_c commute with L_a. So the eigenspaces E^a_λ and E^a_μ must be L_b-invariant and L_c-invariant. So L_b and L_c must be diagonal too and L_a, L_b and L_c commute. From the braid relation it follows $L_bL_cL_b = L_cL_bL_c \Rightarrow L_b = L_c \Rightarrow L_bL_c^{-1} = I \Rightarrow \phi(t_bt_c^{-1}) = I$. Because G' is generated normally by $t_bt_c^{-1}$ it follows that $\phi(G') = \{I\}$.

(2) If L_a only has one eigenvalue λ and the jordan form of $L_a = \lambda I$, then L_a commutes with every other linear representative L_x. Let x be any isotopy class of nonseparating simple closed curves on S. Because a and x are conjugate, L_a and L_x must be conjugate too. So $L_x = FL_aF^{-1} = F\lambda IF^{-1} = \lambda IFF^{-1} = \lambda I$. According to the theorem of Dehn-Lickorish the Mapping Class Group is generated by (finitely many) Dehn twists about nonseparating simple closed curves. So $G = \mathrm{PMod}(S) = \langle t_a \rangle \Rightarrow \phi(G) = \phi(\langle t_a \rangle) = \langle \phi(t_a) \rangle = \langle L_a \rangle$ and therefore the image of ϕ is cyclic. $\Rightarrow \phi$ factors via G'.

(3) If L_a is of the form $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ in some adequate basis, then the eigenspace $E^a_\lambda = \mathrm{span}\{(1,0)\}$ is 1-dimensional. So $L_b = \left(\begin{smallmatrix} \lambda & * \\ 0 & \lambda \end{smallmatrix}\right)$ and $L_c = \left(\begin{smallmatrix} \lambda & * \\ 0 & \lambda \end{smallmatrix}\right)$ because they must preserve E^a_λ and because they are conjugate to L_a they have the same eigenvalues. $L_bL_c = L_cL_b$ because $\left(\begin{smallmatrix} \lambda & a \\ 0 & \lambda \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda & b \\ 0 & \lambda \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda^2 & (a+b)\lambda \\ 0 & \lambda^2 \end{smallmatrix}\right)$. From the braid relation it follows $L_bL_cL_b = L_cL_bL_c \Rightarrow$

$L_b = L_c \Rightarrow L_b L_c^{-1} = I \Rightarrow \phi(t_b t_c^{-1}) = I$. So $\phi(G') = \{I\}$ like in the case with two eigenvalues.

$\square$

LEMMA 11. *Let $F \in \mathrm{PMod}(S)$ and L_x the representative of a Dehn twist along a nonseparating simple closed curve and λ an eigenvalue of L_x. Then:*

$$E_\lambda(FL_xF^{-1}) = F(E_\lambda(L_x)) = F(E_\lambda^x) \tag{3.1}$$

PROOF.

$$\begin{aligned}
F(E_\lambda^x) &= F(\{\vec{v} \in V : \lambda\vec{v} = L_x\vec{v}\}) \\
&= \{F\vec{v} \in V : \lambda\vec{v} = L_x\vec{v}\} \\
&= \{\vec{v} \in V : \lambda F^{-1}\vec{v} = L_xF^{-1}\vec{v}\} \\
&= \{\vec{v} \in V : F\lambda F^{-1}\vec{v} = FL_xF^{-1}\vec{v}\} \\
&= \{\vec{v} \in V : \lambda\vec{v} = FL_xF^{-1}\vec{v}\} \\
&= E_\lambda(FL_xF^{-1})
\end{aligned}$$

$\square$

LEMMA 12 ([**Kor11**], Lemma 4.2). *Let a, b, x and y be four isotopy classes of nonseparating simple closed curves on S, such that there is an orientation preserving homeomorphism*[1] *$f : S \to S$ with $f(x) = a$ and $f(y) = b$. Let λ be an eigenvalue of L_a. Then:*

$$E_\lambda^a = E_\lambda^b \Leftrightarrow E_\lambda^x = E_\lambda^y$$

PROOF. Because of Lemma 1 on Page 13: $ft_xf^{-1} = t_a$ and $ft_yf^{-1} = t_b$, so for the representatives we get $FL_xF^{-1} = L_a$ and $FL_yF^{-1} = L_b$. Furthermore:

$$E_\lambda^a = E_\lambda(L_a) = E_\lambda(FL_xF^{-1})$$

$$E_\lambda^b = E_\lambda(L_b) = E_\lambda(FL_yF^{-1})$$

Then by equation (3.1):

$$E_\lambda(FL_xF^{-1}) = F(E_\lambda^x)$$
$$E_\lambda(FL_yF^{-1}) = F(E_\lambda^y)$$

So we have: $E_\lambda^a = F(E_\lambda^x)$, $E_\lambda^b = F(E_\lambda^y)$, $E_\lambda^x = F^{-1}(E_\lambda^a)$ and $E_\lambda^y = F^{-1}(E_\lambda^b)$. And

$$E_\lambda^a = E_\lambda^b \Leftrightarrow E_\lambda^x = E_\lambda^y$$

$\square$

REMARK 12. *A 3×3 matrix has 6 possible different Jordan forms.* [2]

NOTATION 6. $\mathrm{span}\{(1,0,0)\} := \{(z,0,0) : z \in \mathbb{C}\}$. *More formally:*

$$\mathrm{span}\{\vec{v_1}, \ldots, \vec{v_k}\} := \{z_1\vec{v_1} + \ldots + z_k\vec{v_k} : z_1, \ldots, z_k \in \mathbb{C}\}$$

PROPOSITION 13 ([**Kor11**], Proposition 4.3). *Let S be a surface of genus $g = 2$ and let $\phi : \mathrm{PMod}(S) \to \mathrm{GL}(3, \mathbb{C})$ be a homomorphism. Then*

$$\mathrm{Im}(\phi) \cong \mathbb{Z}_{10}/N$$

where $N \lhd Z_{10}$ is a normal subgroup.

[1] The paper [**Kor11**] says orientation-preserving diffeomorphism. But there is an isomorphism $\pi_0(\mathrm{Homeo}^+(S, \partial S)) \cong \pi_0(\mathrm{Diff}^+(S, \partial S))$.

[2] See The On-Line Encyclopedia of Integer Sequences [**Slo**]. Sequence Number A000219: $1, 1, 3, 6, 13, 24, 48, 86, 160, 282, 500, \ldots$

PROOF. Let $G := \mathrm{PMod}(S)$ and let $G' := [\mathrm{PMod}(S), \mathrm{PMod}(S)]$. Note that $G^{\mathrm{ab}} := G/G' \cong \mathbb{Z}_{10}$. So it suffices to show that $\mathrm{Im}(\phi)$ is abelian, because of the universal property of the abelianisation, any homomorphism from G to an abelian group must factor through G^{ab}, so the image must be a quotient of $\mathbb{Z}_{10}$. Or equivalently it is sufficient to show that $\phi(G') = \{I\}$ because then $G' \subseteq \mathrm{Kern}(\phi)$ and $\mathrm{Im}(\phi) \cong G/\mathrm{Kern}(\phi) \cong G^{\mathrm{ab}}/(\mathrm{Kern}(\phi)/G')$. Let a be an isotopy class of nonseparating simple closed curves on S. Then there are six possible different Jordan forms for L_a:

(1) $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

(2) $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$

(3) $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

(4) $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix}$

(5) $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

(6) $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$

Where $\lambda \neq \mu$, $\mu \neq \nu$ and $\lambda \neq \nu$. Each case should be analysed separately.

- **(1) $L_a = \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$:** Let x be an isotopy class of nonseparating simple closed curves on S. All nonseparating simple closed curves on S are conjugate, so $L_x = F\lambda I F^{-1} = \lambda I$. Because x was free to choose, every dehn twist about a nonseparating simple closed curve on S is represented by λI. Because $\mathrm{PMod}(S)$ is generated by such dehn twists, it follows that $\phi(G) = \langle \lambda I \rangle$ is cyclic.
- **(2) $L_a = \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{smallmatrix}\right)$:** We have $E^a_\lambda = \mathrm{Kern}(L_a - \lambda I) = \mathrm{span}\{(1,0,0)\}$. And analogously for the two other eigenvalues. So take two isotopy classes of nonseparating closed curves b and c intersecting at one point such that $b \cup c$ is disjoint from a. L_b and L_c must leave each eigenspace of L_a invariant. So they must be both diagonal $\left(\begin{smallmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{smallmatrix}\right)$ and so L_b and L_c commute. Because they intersect once the braid relation is valid and so: $L_bL_cL_b = L_cL_bL_c \Rightarrow L_b = L_c$. From this it follows like before in the proof for 2×2 matrices, that $\phi(L_b) = \phi(L_c) \Rightarrow \phi(L_bL_c^{-1}) = I$ and because $G' = \left\langle t_bt_c^{-1} \right\rangle^G \Rightarrow \phi(G') = \{I\}$.
- **(3) $L_a = \left(\begin{smallmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$:** We have $E^a_\lambda = \mathrm{Kern}(L_a - \lambda I) = \mathrm{span}\{(1,0,0)\}$. Let b and c be two isotopy classes of nonseparating simple closed curves in S like in the case (2) above, such that b and c intersect once, and both are disjoint from a. L_b and L_c must preserve E^a_λ, so must be of the form $\left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right)$. Because generalised eigenspaces must be preserved too if follows $\mathrm{Kern}((L_a - \lambda I)^2) = \mathrm{span}\{(1,0,0),(0,1,0)\}$ must be preserved and so L_b and L_c must be of the form $\left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{smallmatrix}\right)$. Because they are also conjugate to

L_a they must have the same eigenvalues and so are of the form:

$$\begin{pmatrix} \lambda & * & * \\ 0 & \lambda & * \\ 0 & 0 & \lambda \end{pmatrix}$$

Because of Lemma 2 and from the braid relation we get $L_bL_cL_b = L_cL_bL_c \Rightarrow L_b = L_c \Rightarrow \phi(G') = \{I\}$.

(4) $L_a = \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{smallmatrix}\right)$**:** For the eigenspaces we get $E^a_\lambda = \operatorname{span}\{(1,0,0)\}$ and $E^a_\mu = \operatorname{span}\{(0,1,0)\}$. Let b and c again be two isotopy classes of nonseparating simple closed curves on S such that b and c intersect once, and they both are disjoint from a. The eigenspaces of L_a must be preserved by L_b and L_c so:

$$L_b, L_c = \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

Because of the braid relation:

$$\begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \cdot \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}^2 c_{11} & 0 & b_{11}b_{13}c_{11} + b_{33}\left(b_{11}c_{13} + b_{13}c_{33}\right) \\ 0 & b_{22}^2 c_{22} & b_{22}b_{23}c_{22} + b_{33}\left(b_{22}c_{23} + b_{23}c_{33}\right) \\ 0 & 0 & b_{33}^2 c_{33} \end{pmatrix}$$

and

$$\begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \cdot \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} c_{11}^2 & 0 & b_{11}c_{11}c_{13} + \left(b_{13}c_{11} + b_{33}c_{13}\right) c_{33} \\ 0 & b_{22} c_{22}^2 & b_{22}c_{22}c_{23} + \left(b_{23}c_{22} + b_{33}c_{23}\right) c_{33} \\ 0 & 0 & b_{33} c_{33}^2 \end{pmatrix}$$

must be equal. So: $b_{11} = c_{11}$, $b_{22} = c_{22}$ and $b_{33} = c_{33}$. Then from $L_aL_b = L_bL_a$ and $L_aL_c = L_cL_a$ we get:

$$\begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda b_{11} & 0 & \mu b_{13} \\ 0 & \mu b_{22} & b_{22} + \mu b_{23} \\ 0 & 0 & \mu b_{33} \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} = \begin{pmatrix} \lambda b_{11} & 0 & \lambda b_{13} \\ 0 & \mu b_{22} & \mu b_{23} + b_{33} \\ 0 & 0 & \mu b_{33} \end{pmatrix}$$

So $b_{22} = b_{33} = c_{22} = c_{33}$ and $b_{13} = c_{13} = 0$. Because L_a, L_b and L_c must have the same eigenvalues it follows that:

$$L_b, L_c = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & * \\ 0 & 0 & \mu \end{pmatrix}$$

But matrices of this form commute. So again from the braid relation $L_bL_cL_b = L_cL_bL_c \Rightarrow L_b = L_c$ and like in the cases before $\phi(G') = \{I\}$.

(5) $L_a = \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$**:** For the eigenspace we get $E^a_\lambda = \operatorname{span}\{(1,0,0),(0,1,0)\}$. Let b be the isotopy class of nonseparating closed simple curves on S such that b intersects a in one point ($i(a,b) = 1$). There are two cases to check.

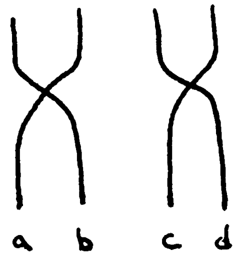

FIGURE 3. Intersection of curves for Proposition 13 Part (5a)

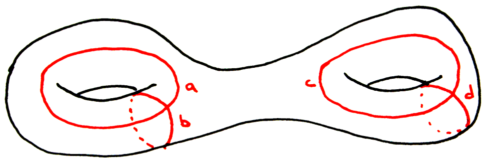

FIGURE 4. Example of how the intersection of curves for Proposition 13 Part (5a) could look like in a genus 2 surface.

(a) $E^a_\lambda \neq E^b_\lambda$**:** Let c and d be two isotopy classes of nonseparating simple closed curves on S such that c intersects d at one point and c and d are disjoint from both a and b. See Figure 3 and Figure 4 as example. Then $E^a_\lambda \cap E^b_\lambda$ is of dimension 1 and E^a_λ is of dimension 2. Both must be invariant under L_c and L_d and L_c and L_d must have the same eigenvalues with multiplicities as L_a and so L_c and L_d must be of the form:

$$L_c, L_d = \begin{pmatrix} \lambda & * & * \\ 0 & \lambda & * \\ 0 & 0 & \lambda \end{pmatrix}$$

Again from Lemma 2 and from the braid relation we get $L_cL_dL_c = L_dL_cL_d \Rightarrow L_c = L_d \Rightarrow \phi(G') = \{I\}$.

(b) $E^a_\lambda = E^b_\lambda$**:** Let b' be any isotopy class of nonseparating simple closed curves on S intersecting a in one point. Then from Lemma 12 we get that $E_b = E'_b$. Now let c be any isotopy class of simple closed nonseparating curves on S. Then by Theorem 4 there exists a sequence of isotopy classes of nonseparating simple closed curves $a = a_0, a_1, \dots, a_k = c$ such that $i(a_i, a_{i-1} \forall i \in \{1, \dots, k\}$ and therefore $E^a_\lambda = E^{a_0}_\lambda = E^{a_1}_\lambda = \dots = E^{a_k}_\lambda = E^c_\lambda$ because $G = \mathrm{PMod}(S)$ is generated by dehn twists about isotopy classes of nonseparating simple closed curves on S, E_a must be invariant under $L \in \phi(G)$ because L can be written as $L = L_{b_0}L_{b_1}\dots L_{b_m}$ for some isotopy classes of nonseparating simple closed curves in S $b_0, \dots, b_m$. So there must be a homomorphism $\bar{\phi} \to \mathrm{GL}(E^a_\lambda) = \mathrm{GL}(2,\mathbb{C})$. Think of this as:

$$G \xrightarrow{\phi} \mathrm{GL}(3,\mathbb{C}) \xrightarrow{\left(\begin{smallmatrix} a & b & * \\ d & e & * \\ * & * & * \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \mathrm{GL}(2,\mathbb{C})$$

With Proposition 12 it follows that $\bar{\phi}(G)$ is cyclic and so for any element of the commutator subgroup $f \in G' : \bar{\phi}(f) = I$, therefore

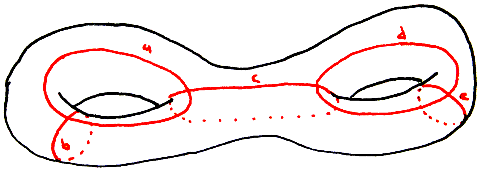

FIGURE 5. Example of how the intersection of curves for Proposition 13 Part (6a) could look like in a genus 2 surface.

the matrix of $\phi(f)$ must have the form:

$$\phi(f) = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix}$$

$\phi(G')$ consists only of upper triangular matrices, and so by Lemma 5 is solvable. The group G' is perfect by Theorem 7 and so by Lemma 7, any homomorphism $G' \longrightarrow \phi(G')$ is trivial.

(6) $L_a = \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{smallmatrix}\right)$**:** For the λ-eigenspace of L_a we get $E^a_\lambda = \text{span}\{(1,0,0),(0,1,0)\}$. Let b be the isotopy class of nonseparating closed simple curves on S such that b intersects a in one point ($i(a,b) = 1$). Again there are two cases to check:

(a) $E^a_\lambda \neq E^b_\lambda$**:** By Lemma 12 it follows that for any isotopy classes x and y of nonseparating simple closed curves in S intersecting at one point ($i(x,y) = 1$): $E^x_\lambda \neq E^y_\lambda$. So let c, d and e be isotopy classes of nonseparating simple closed curves on S. Such that $i(b,a) = i(a,c) = i(c,d) = i(d,e) = 1$ and they are otherwise disjoint. See Figure 5 for an example. Let $(\vec{v_1}, \vec{v_2}, \vec{v_3}) : \vec{v_1} \in E^a_\lambda \cap E^b_\lambda, \vec{v_2} \in E^a_\lambda, \vec{v_3} \in E^a_\mu$ so that it is a basis. $E^a_\lambda \cap E^b_\lambda$, E^a_λ and E^a_μ must be L_d and L_e invariant. So:

$$L_d, L_e = \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

From the Braid relation $L_d L_e L_d = L_e L_d L_e$ it follows:

$$\begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \cdot \begin{pmatrix} e_{11} & e_{12} & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{pmatrix} \cdot \begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$$

$$= \begin{pmatrix} d_{11}^2 e_{11} & d_{11} d_{12} e_{11} + d_{22}\left(d_{11} e_{12} + d_{12} e_{22}\right) & 0 \\ 0 & d_{22}^2 e_{22} & 0 \\ 0 & 0 & d_{33}^2 e_{33} \end{pmatrix}$$

and

$$\begin{pmatrix} e_{11} & e_{12} & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{pmatrix} \cdot \begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \cdot \begin{pmatrix} e_{11} & e_{12} & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{pmatrix}$$

$$= \begin{pmatrix} d_{11} e_{11}^2 & d_{11} e_{11} e_{12} + \left(d_{12} e_{11} + d_{22} e_{12}\right) e_{22} & 0 \\ 0 & d_{22} e_{22}^2 & 0 \\ 0 & 0 & d_{33} e_{33}^2 \end{pmatrix}$$

So $d_{11} = e_{11}$, $d_2 = e_{22}$ and $d_{33} = e_{33}$. Because L_d and L_e must have the same eigenvalues as L_a, the following three forms are possible: $\left(\begin{smallmatrix} \lambda & * & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mu & * & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \lambda & * & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$:

(i) $L_d, L_e = \left(\begin{smallmatrix} \lambda & * & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{smallmatrix}\right)$**:** But in this case

$$\begin{pmatrix} \lambda & d_{12} & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} \lambda & e_{12} & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda & e_{12} & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} \lambda & d_{12} & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^2 & \lambda d_{12} + \lambda e_{12} & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \mu^2 \end{pmatrix}$$

So L_d and L_e commute, and now again from the braid relation $L_dL_eL_d = L_eL_dL_e$ we get $L_d = L_e$ and $\phi(G') = \{I\}$.

(ii) $L_d, L_e = \left(\begin{smallmatrix} \mu & * & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$ **or** $L_d, L_e = \left(\begin{smallmatrix} \lambda & * & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$**:** If $L_d, L_e = \left(\begin{smallmatrix} \lambda & * & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$ then $E^d_\lambda = E^e_\lambda$ a contradiction. If $L_d, L_e = \left(\begin{smallmatrix} \mu & * & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$ then must $E^e_\mu = \mathrm{span}\{(1,0,0)\}$ be L_c invariant, and so:

$$L_c = \begin{pmatrix} c_{11} & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

From the braid relation $L_aL_cL_a = L_cL_aL_c$ we get:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^2 c_{11} & \lambda^2 c_{12} & \lambda\mu c_{13} \\ 0 & \lambda^2 c_{22} & \lambda\mu c_{23} \\ 0 & \lambda\mu c_{32} & \mu^2 c_{33} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda c_{11}^2 & \lambda c_{11}c_{12} + \lambda c_{22}c_{12} + \mu c_{13}c_{32} & \lambda c_{11}c_{13} + \mu c_{33}c_{13} + \lambda c_{12}c_{23} \\ 0 & \lambda c_{22}^2 + \mu c_{23}c_{32} & \lambda c_{22}c_{23} + \mu c_{33}c_{23} \\ 0 & \lambda c_{22}c_{32} + \mu c_{33}c_{32} & \mu c_{33}^2 + \lambda c_{23}c_{32} \end{pmatrix}$$

And therefore $c_{11} = \lambda$. From the Braid relation $L_cL_dL_c = L_dL_cL_d$ we get:

$$\begin{pmatrix} \lambda & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix} \cdot \begin{pmatrix} \mu & d_{12} & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} \lambda & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^2\mu & \lambda\mu c_{12} + \lambda c_{13}c_{32} + c_{22}\left(\lambda c_{12} + \lambda d_{12}\right) & \lambda\mu c_{13} + \lambda c_{33}c_{13} + c_{23}\left(\lambda c_{12} + \lambda d_{12}\right) \\ 0 & \lambda c_{22}^2 + \lambda c_{23}c_{32} & \lambda c_{22}c_{23} + \lambda c_{33}c_{23} \\ 0 & \lambda c_{22}c_{32} + \lambda c_{33}c_{32} & \lambda c_{33}^2 + \lambda c_{23}c_{32} \end{pmatrix}$$

$$= \begin{pmatrix} \mu & d_{12} & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} \lambda & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix} \cdot \begin{pmatrix} \mu & d_{12} & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda\mu^2 & \lambda\mu d_{12} + \lambda\left(\mu c_{12} + c_{22}d_{12}\right) & \lambda\left(\mu c_{13} + c_{23}d_{12}\right) \\ 0 & \lambda^2 c_{22} & \lambda^2 c_{23} \\ 0 & \lambda^2 c_{32} & \lambda^2 c_{33} \end{pmatrix}$$

But then $\lambda = \mu$ a contradiction.

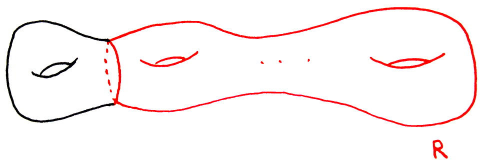

FIGURE 6. Explanation of surfaces for Lemma 13

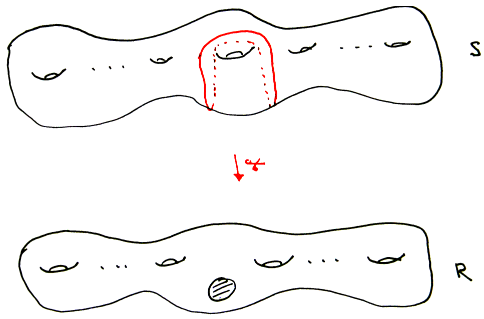

FIGURE 7. Example how a torus can be cut out so that the rest surface has one boundary component.

(b) $E^a_\lambda = E^b_\lambda$**:** Let b' be any isotopy class of nonseparating simple closed curves on S intersecting a in one point. Then from Lemma 12 we get that $E_b = E'_b$. Now let c be any isotopy class of simple closed nonseparating curves on S. Then by Theorem 4 there exists a sequence of isotopy classes of nonseparating simple closed curves $a = a_0, a_1, \ldots, a_k = c$ such that $i(a_i, a_{i-1} \forall i \in \{1, \ldots, k\}$ and therefore $E^a_\lambda = E^{a_0}_\lambda = E^{a_1}_\lambda = \ldots = E^{a_k}_\lambda = E^c_\lambda$ because $G = \mathrm{PMod}(S)$ is generated by dehn twists about isotopy classes of nonseparating simple closed curves on S, E_a must be invariant under $L \in \phi(G)$ because L can be written as $L = L_{b_0} L_{b_1} \ldots L_{b_m}$ for some isotopy classes of nonseparating simple closed curves in S $b_0, \ldots, b_m$. So there must be a homomorphism $\bar{\phi} \to \mathrm{GL}(E^a_\lambda) = \mathrm{GL}(2, \mathbb{C})$. Think of this as:

$$G \xrightarrow{\phi} \mathrm{GL}(3, \mathbb{C}) \xrightarrow{\left(\begin{smallmatrix} a & b & * \\ d & e & * \\ * & * & * \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \mathrm{GL}(2, \mathbb{C})$$

With Proposition 12 it follows that $\bar{\phi}(G)$ is cyclic. And so for any element of the commutator subgroup $f \in G' : \bar{\phi}(f) = I$, so the matrix of $\phi(f)$ must have the form:

$$\phi(f) = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix}$$

So $\phi(G')$ consists only of upper triangular matrices, and so by Lemma 5 is solvable. The group G' is perfect by Theorem 7. And so by Lemma 7, any homomorphism $G' \longrightarrow \phi(G')$ is trivial.

$\square$

LEMMA 13 ([**Kor11**]). *Let S be a surface of genus $g \geq 3$ and let $n \leq 2g - 1$. Let R be a subsurface of S homeomorphic to a compact connected surface of genus $g - 1$ with one boundary component. Imagine this as the rest of the surface S that* *is remaining after cutting away one torus. See Figures 6 and 7. The embedding of R in S induces an embedding of* $\mathrm{PMod}(R)$ *in* $\mathrm{PMod}(S)$*:*

$$f \mapsto f'(x) = \begin{cases} f(x) & \textit{if } x \in R \\ x & \textit{if } x \in S - R \end{cases}$$

Let $G := \mathrm{PMod}(S)$, $G_R := \mathrm{PMod}(R)$, $G'_R := [\mathrm{PMod}(R), \mathrm{PMod}(R)]$ *and* $G_H^{ab} := G_R/G'_R$*. If there exists a* $\phi(G_R)$ *invariant subspace* $V \subseteq \mathbb{C}^n$ *of dimension* $r := \dim(V)$ *such that* $2 \leq r \leq n-2$*. Then* ϕ *induces two homomorphisms:*

- $\phi_1 : G_R \to \mathrm{GL}(V) \cong \mathrm{GL}(r, \mathbb{C})$ *and*
- $\phi_2 : G_R \to \mathrm{GL}(\mathbb{C}^n/V) \cong \mathrm{GL}(n-r, \mathbb{C})$

If $\phi_1(G_R)$ *and* $\phi_2(G_R)$ *both are cyclic then* $\phi(G)$ *is trivial.*

PROOF. Let a and b be two isotopy classes of nonseparating simple closed curves on R, with $i(a,b) = 1$. $\phi_1(G_R)$ and $\phi_2(G_R)$ are both cyclic, so they are also abelian. So with Lemma 9 it follows that $\phi_1(t_a) = \phi_1(t_b) \Rightarrow \phi_1(t_a t_b^{-1}) = I$ and $\phi_2(t_a) = \phi_2(t_b) \Rightarrow \phi_2(t_a t_b^{-1}) = I$. G'_R is generated normally by elements of the form $t_a t_b^{-1}$. So

$$\phi_1(G'_R) = \phi_1(\langle t_a t_b^{-1} \rangle^{G'_R}) = \langle \{\phi_1(g)\phi_1(t_a t_b^{-1})\phi_1(g^{-1}) : g \in G'_R\} \rangle = \langle I \rangle = I$$

and

$$\phi_2(G'_R) = \phi_2(\langle t_a t_b^{-1} \rangle^{G'_R}) = \langle \{\phi_2(g)\phi_2(t_a t_b^{-1})\phi_2(g^{-1}) : g \in G'_R\} \rangle = \langle I \rangle = I$$

Thus for all $f \in G'_R$ there is some basis of $\mathbb{C}$ such that $\phi(f)$ has the form:

$$\phi(f) = \begin{array}{c} \overbrace{\hphantom{1 \quad 1 \quad \ddots \quad 1}}^{\phi_1(G'_R)=I_r} \hphantom{\quad * \quad \cdots \quad *} \\ \left(\begin{array}{cccc|cccc} 1 & & & & * & \cdots & & * \\ & 1 & & & \vdots & \ddots & & \vdots \\ & & \ddots & & & & & \\ & & & 1 & * & \cdots & & * \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) \\ \underbrace{\hphantom{1 \quad 1 \quad \ddots \quad 1}}_{\phi(G'_R)V \subseteq V} \quad \underbrace{\hphantom{* \quad \cdots \quad *}}_{\phi_2(G'_R)=I_{n-r}} \end{array} = \begin{pmatrix} I_n & F \\ 0 & I_{n-r} \end{pmatrix}$$

In the matrix it is indicated with braces where the various entries come from. The subgroup of matrices of this form is abelian. Therefore it is also solvable, and because G'_R is perfect it follows from Lemma 7 that $\phi(G'_R)$ is trivial, and so $\phi(ab^{-1}) = I$. Because of Theorem 6 by Powell [**Pow78**] the mapping class group for $g \geq 3$ is perfect, so $G = G'$ but since this group is generated normally by $t_a t_b^{-1}$ it follows that

$$\phi(G) = \phi(\langle t_a t_b^{-1} \rangle^G) = \phi(\langle \{f t_a t_b^{-1} f^{-1} : f \in G\} \rangle) = \langle \phi(\{f t_a t_b^{-1} f^{-1} : f \in G\}) \rangle$$

$$= \langle \{F \phi(t_a t_b^{-1}) F^{-1} : F \in \phi(G)\} \rangle = \langle \{F I F^{-1} : F \in \phi(G)\} \rangle = I$$

. $\square$

This concludes the preparations for proofing the main theorem.

3.2. Proof of the Main Theorem.

Proof of Theorem 3, [**Kor11**]. Remember that the conditions for the theorem were: $g \geq 1$ and $n \leq 2g-1$.

$g = 1$ **and** $n = 1$**:** In this case we have a map $\phi : \mathrm{PMod}(S) \longrightarrow \mathrm{GL}(1, \mathbb{C})$ But this is the same as $\phi : \mathrm{PMod}(T^2) \cong \mathrm{SL}(2, \mathbb{Z}) \longrightarrow \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$. $\mathbb{C}^*$ is abelian, so by the universal property of the abelianisation any homomorphism must factor through the abelianized group:

$$\begin{tikzcd} \mathrm{PMod}(S) \arrow[r] \arrow[d] & \mathbb{C}^* \\ \mathrm{PMod}(S)^{\mathrm{ab}} \arrow[ur] & \end{tikzcd}$$

From Theorem 8 it follows that $\phi(G) = \mathbb{Z}_{12}/N$ if $q = 0$ and $\phi(G) = \mathbb{Z}^q/N$ if $q \geq 1$ where $N \triangleleft \mathbb{Z}_{12}$ respectively $N \triangleleft \mathbb{Z}^q$ is a normal subgroup.

$g = 2$ **and** $n \leq 3$**:** This has been proved in Proposition 12 and Proposition 13.

$g \geq 3$**:** Proof by induction. Assume the Theorem holds for all surfaces with genus $g-1$. There is an isomorphism from $GL(k-1, \mathbb{C})$ to a subgroup of $GL(k, \mathbb{C})$. e.g. given by:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(k-1)} \\ a_{21} & a_{22} & \cdots & a_{2(k-1)} \\ \vdots & & & \vdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & a_{(k-1)(k-1)} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(k-1)} & 0 \\ a_{21} & a_{22} & \cdots & a_{2(k-1)} & 0 \\ \vdots & & & \vdots & 0 \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & a_{(k-1)(k-1)} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

So it is sufficient to proof the theorem for $n = 2g-1$.

Let R be a subsurface of S homeomorphic to a compact connected surface of genus $g-1$ with one boundary component. Imagine this as the rest of the surface S that is remaining after cutting away one torus. See Figures 6 and 7. The embedding of R in S induces an embedding of $\mathrm{PMod}(R)$ in $\mathrm{PMod}(S)$:

$$f \mapsto f'(x) = \begin{cases} f(x) & \text{if } x \in R \\ x & \text{if } x \in S - R \end{cases}$$

Then set $G := \mathrm{PMod}(S)$, $G_R := \mathrm{PMod}(R)$, $G'_R := [\mathrm{PMod}(R), \mathrm{PMod}(R)]$ and $G_H^{\mathrm{ab}} := G_R/G'_R$. Let a and b be two isotopy classes of nonseparating simple closed curves on S with $i(a, b) = 1$ such that $a \cup b$ is disjoint from R. If there is a subspace $V \subseteq \mathbb{C}^n$ of dimension r with $2 \leq r \leq n-2$ which is a direct sum of eigenspaces of L_a then V is $\phi(G_R)$ invariant. And therefore by Lemma 13 and by the assumption that the theorem holds (meaning the image of any $G_R \longrightarrow \mathrm{GL}(n, \mathbb{C})$ is cyclic) for all surfaces of genus $g-1$ it follows that $\phi(G) = I$. There exists such a subspace if L_a has at least three distinct eigenvalues: Note, that $\dim(E_\lambda^a) \geq 1$ for each eigenspace and therefore if L_a has at least three distinct eigenvalues $\dim(E_\lambda^a) \leq n-2$ for each eigenspace and $\dim(V) \geq 2$.

If on the other hand, there is no such subspace V. Then L_a has at most two eigenvalues and each eigenspace of L_a is either 1-, $(n-1)$- or n-dimensional. Thus, the jordan form of L_a must have one of the following forms:

(1)

$$\lambda I_n = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 0 & \lambda & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \lambda & 0 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

(2)

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \ddots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

(3)

$$\lambda I_{n-2} \oplus J_{\lambda,2} = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

(4)

$$\lambda I_{n-1} \oplus \mu I_1 = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \lambda & 0 \\ 0 & 0 & \cdots & 0 & \mu \end{pmatrix}$$

Choose a basis so that the matrix of L_a is in jordan form. Each case is analysed separately:

$L_a = \lambda I_n$: In this case for any isotopy class c of nonseparating simple closed curves on S c is conjugate to a and therefore: $L_c = L_a \Rightarrow L_c L_a^{-1} = I$. Because G is generated by finitely many dehn twists about nonseparating simple closed curves it follows that:

$$\phi(G) = \phi(\langle t_1, t_2, \ldots \rangle) = \langle \lambda I \rangle$$

But then $\phi(G)$ is cyclic, but because $(\lambda I)^{-1} \in \langle \lambda I \rangle$, $\lambda = 1$, so the image of ϕ is trivial.

$L_a = J_{\lambda,n}$: Here $\dim(\mathrm{Kern}(L_a - \lambda I)) = \dim(\mathrm{span}\{(1,0,\ldots)\}) = 1$, but $\dim(\mathrm{Kern}((L_a - \lambda I)^2)) = \dim(\mathrm{span}\{(1,0,\ldots),(0,1,0,\ldots)\}) = 2$, and because $\mathrm{Kern}((L_a - \lambda I)^2)$ must be $\phi(G_R)$ invariant it follows by lemma 13 and the assumption that the theorem holds for all surfaces of genus $g-1$ that $\phi(G)$ is trivial.

$L_a = \lambda I_{n-2} \oplus J_{\lambda,2}$: In this case $\dim(\mathrm{Kern}(L_a - \lambda I)) = n-1$ therefore the eigenspace of L_b must also be of dimension $n-1$. If $E^a_\lambda \neq E^b_\lambda$, then $E^a_\lambda \cap E^b_\lambda$ is of dimension $n-2$ and $\phi(G_R)$-invariant. By Lemma 13 and the assumption that the theorem holds for all surfaces of genus $g-1$

it follows that $\phi(G)$ is trivial. If $E_\lambda^a = E_\lambda^b$, then from lemma 12 it follows that $E_\lambda^c = E_\lambda^d$ for any two isotopy classes c and d of nonseparating simple closed curves on S intersecting in exactly one point. From Theorem 4 it follows that there is a sequence of curves between a and any other isotopy class a' of nonseparating simple closed curves on S intersecting each predecessor and successor in in exactly one point. Therefore $E_\lambda^a = E_\lambda^{a'}$. Because G is generated by finitely many dehn twists about isotopy classes of nonseparating simple closed curves on S it follows that for every mapping class $f \in \mathrm{PMod}(S)$, the representation of f, $\phi(f)$ must be in upper triangular form. But by Lemma 5, the subgroup $\Delta\mathrm{GL}(n, \mathbb{C})$ of $\mathrm{GL}(n, \mathbb{C})$ consisting of all upper triangular matrices is solvable. And because G is perfect by Theorem 6 it follows with Lemma 7 that $\phi(G)$ is trivial.

$L_a = \lambda I_{n-1} \oplus \mu I_1$: Exactly the same proof as for the case with $L_a = \lambda I_{n-2} \oplus J_{\lambda,2}$.

□

CHAPTER 3

Epilogue

Only that day dawns to which
we are awake.
There is more day to dawn.
The sun is but a morning star.
– *Henry David Thoreau*

1. Outlook

The study of mapping classes, Dehn twists and linear representations thereof has a wealth of applications in mathematics and physics. This section gives a short outlook of topics related to the study of mapping class groups. As this is only a short overview, definitions will not be given.

For example mapping class groups are related to braid groups. Classical Artin braid groups [**Art47**] are just a special kind of mapping class groups. Braids also appear in category theory.

There are important connections between the mapping class groups and Teichmüller spaces [**FM12**].

In an analogous way to the Jordan canonical form, there is a similar classification of elements of mapping class groups called the Nielsen-Thurston Classification [**FM12**].

In physics there are applications in conformal field theory [**Gan07**], quantum field theory [**Buf03**] and in string theory [**Nag95**].

2. About the creation of this thesis

Creating this thesis was certainly one of the most interesting part so far of studying mathematics. Even though at the beginning it looked like a difficult task and the amount of information to research and learn seemed like a lot and keeping track of what is important and what is not was not easy but after the initial difficulty of learning the new theories and learning the new mathematical vocabulary was conquered, things were gradually getting easier, and it was possible to relate the newly learned knowledge to the already present mathematical knowledge and the whole task didn't look so complicated anymore. After finalizing an initial pre-draft of the thesis on paper, most of the work that remained and was time consuming was typesetting the text in LaTeX on the computer and drawing and scanning the graphics needed in the text. An initial attempt to create the graphics using the latex package ps-tricks or using a vector graphics program was quickly abandoned as it was deemed to be too time consuming. So graphics were created by hand using a technical-pen, then scanned, enhanced using photo edit software and finally converted to png. ImageMagick[1] was used to convert the pictures to eps, but I realised later that Pdflatex doesn't support eps, and because I only needed pdf output and no dvi or ps, I kept all pictures in png format only. Based on the

[1]http://www.imagemagick.org

commonly used scientific writing style guidelines, I tried to avoid using the first-person (singular) throughout the writing of the text, except in the preface and this epilogue. The Chicago Manual of Style [**Uni10**] seems to be one of the most often used style guides in scientific writing, but I only glanced through it quickly as I had already written the greatest part of my thesis when I heard about it.

APPENDIX A

Additional Definitions and Conventions used throughout the text

Let no man who is not a Mathematician read the elements of my work.

– *Leonardo da Vinci*

1. Set theoretic conventions

2. Some useful algebraic definitions and properties

DEFINITION 22. *A sequence of groups G_i and homomorphisms f_j:*

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} G_n$$

*is called **exact** if for each $i \in \{1, \dots, n-1\} : \mathrm{Im}(f_i) = \mathrm{Kern}(f_{i+1})$. An exact sequence is called **short** if it is of the form:*

$$1 \to G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \to 1$$

3. Algebraic conventions and Notations

DEFINITION 23 ([**Lan02**, **Rot02**, **Art11**]). *If $g \in G$ is a generator of a group. Then $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$ denotes the group generated by this element using the group operation. If S is a generating set then the group **generated** by it is denoted in the same way $\langle S \rangle$ (the intersection of all subgroups of G containing S):*

$$\langle S \rangle = \bigcap_{S \subseteq H \subseteq G} H$$

*Relations may be indicated using the notation $\langle S | R \rangle$. $\langle S \rangle^G$ denotes the group **normally generated** (also called normal closure) by S (the intersection of all normal subgroups of G containing S):*

$$\langle S \rangle^G = \bigcap_{S \subseteq N \trianglelefteq G} N$$

$\langle S \rangle^G$ is the group generated by all conjugate elements of S: $\langle S \rangle^G = \left\langle \{gSg^{-1} : g \in G\} \right\rangle$.

4. Fundamental polygons

Every closed surface can be constructed from an even-sided oriented polygon called **fundamental polygon** by gluing together pairwise identified edges.[**Mun00**] Fundamental polygons are useful in visualizing surfaces and in showing the action of some mapping classes on the surface.

EXAMPLE 7. *Figure 1 shows examples of fundamental polygons for the sphere, the torus and the klein bottle.*

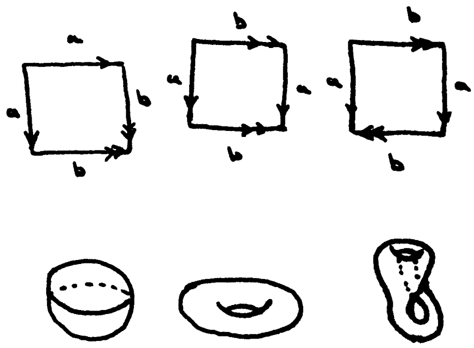

FIGURE 1. Examples of fundamental polygons for the sphere, the torus and the klein bottle

REMARK 13. *Instead of drawing the polygon, it can also be represented by a string of distinct letters representing each edge where identified edges use the same letter and an exponent of* 1 *or* −1 *indicates the direction of the edge. The string is read from the polygon by reading the edges in clockwise direction, and putting exponent of* −1 *on edges going in the opposite direction.*

EXAMPLE 8. *The string representation of the fundamental polygon for the sphere, torus and klein bottle as given in the previous example are given here:*

- *Sphere:* $abb^{-1}a^{-1}$
- *Torus:* $bab^{-1}a^{-1}$
- *Klein Bottle:* $baba^{-1}$

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