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Undistortedness of Lipschitz n-connected closed subsets in quasi-convex metric spaces of finite Assouad-Nagata dimension

Master Thesis

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Abstract

Given a geodesic metric space X and a non-empty closed subset $Z \subset X$, Young showed in [You14] that undistortedness of Z can be shown given that the Assouad-Nagata-dimension of X is finite and Z is Lipschitz n-connected. This improves upon a previous result by Lang and Schlichenmaier in [LS05]. The aim of this thesis is to elaborate on the proof of Young's theorem, fill in missing pieces whenever possible and explore some applications of the theorem. During the work some errors in the article of Young have been found and attempts to correct those errors have been made. Young himself fixed the last part of the proof in an Erratum [You15], these corrections have been included in the thesis. It was possible to make some small improvements to Young's result, those have been added to the text.

Dedicated to my wife Sara.

Acknowledgements

> The best of ideas is hurt by uncritical acceptance and thrives on critical examination.
>
> George Pólya

I would like to thank Prof. Urs Lang for the helpful suggestions and advice while writing this thesis. Furthermore I'd like to thank the LaTeX Stack Exchange community[1] for useful instructions regarding the use of PGF/TikZ[2], which I used to draw most of the graphics that appear in this text.

[1] http://tex.stackexchange.com/

[2] http://pgf.sourceforge.net/

Preface

I am interested in mathematics only as a creative art.

Godfrey Harold Hardy

This thesis was typeset using LaTeX. A heavily modified version of the thesis template found on the ETH website of the Center for Algorithms, Discrete Mathematics and Optimization [3] was used as a starting point. Citations were managed using the open source software Zotero [4] together with the **biblatex** package[5] and biber tool[6]. I used revision control using Mercurial[7] and the TortoiseHG[8] software together with bitbucket.org[9] for backup and synchronisation between my main computer and laptop. I created most of the graphics using the LaTeX package PGF/TikZ[10].

[3] `http://www.cadmo.ethz.ch/education/thesis/template`
[4] `https://www.zotero.org/`
[5] `https://www.ctan.org/pkg/biblatex`
[6] `https://www.ctan.org/pkg/biber`
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[8] `http://tortoisehg.bitbucket.org/`
[9] `https://www.bitbucket.org`
[10] `https://www.ctan.org/pkg/pgf?lang=en`

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Part I

Introduction and Definitions

Chapter 1

Introduction

You can't take three from two,
Two is less than three,
So you look at the four in the tens place.
Now that's really four tens
So you make it three tens,
Regroup, and you change a ten to ten ones,
And you add 'em to the two and get twelve,
And you take away three, that's nine.
Is that clear?

Tom Lehrer

The main parts of this thesis are contained in the following two chapters. Chapter 2 focuses on introducing the concepts, definitions and theorems, as well as stating the main theorem. In Chapter 3 the proof of the main theorem is then carried out.

This thesis is based on the article "Lipschitz connectivity and filling invariants in solvable groups and buildings" by Robert Young [You14] in which he showed that under certain conditions, a closed subset of a metric space is undistorted given that the Assouad-Nagata-dimension of said space is finite.

Before properly defining everything in Chapter 2, this introduction will give a short overview of all the important concepts that are used further on.

An important role in the thesis plays the notion of *Lipschitz functions* between metric spaces. That is, functions $f : X \to Y$ for which there exists a constant $L > 0$, such that

$$d(f(x), f(y)) \leq L \cdot d(x, y),$$

for any two points x, y in X. We use $\mathrm{Lip}(f)$ to denote the least such constant. Lipschitz functions constitute an important class of functions in metric geometry.

The main theorem deals with metric spaces of finite Assouad-Nagata-dimension. This definition of dimension was studied by Patrice Assouad in [Ass]. The *Assouad-Nagata-dimension* of a metric space is defined as the smallest integer $n \geq 0$ for which there exists a constant $c > 0$ such that for all $d > 0$ there exists a covering of the space by sets with diameter bounded by cd, and such that any set with diameter less than or equal to d meets at most $n + 1$ sets in the cover.

Another important concept required for the theorem is the property of Lipschitz n-connectedness. A metric space X is called *Lipschitz n-connected* if there exists a constant $c > 0$ such that for any $0 \leq d \leq n$ one can extend any Lipschitz map $f : S^d \to X$ to $\bar{f} : D^{d+1} \to X$ such that $\mathrm{Lip}(\bar{f}) \leq c\,\mathrm{Lip}(f)$. Here S^d and D^{d+1} denote the unit sphere and closed unit ball in $\mathbb{R}^{d+1}$.

In the course of describing and proving the theorem one deals with a homology theory based on Lipschitz chains. A *Lipschitz d-chain in X* is just a formal sum of Lipschitz maps $\Delta^d \to X$. Of particular interest will be Lipschitz chains on a simplicial complex constructed as the nerve of a cover. The *nerve of a cover* is basically the simplicial complex constructed by taking a vertex for each set in the cover, an edge for any two intersecting sets and higher dimensional simplicies for higher number of intersections in the cover.

By Rademacher's theorem[1], Lipschitz chains are differentiable almost everywhere, and therefore one can assign a volume element $V(x)$ to each point x of Δ^d of some Lipschitz map $\Delta^d \to X$ by sending an orthonormal basis at x along $D_x f$ and taking the volume of the resulting implied parallelepiped in the tangent space $T_x X$. One can then define the *mass* of a Lipschitz chain α as the integral

$$\mathrm{mass}(\alpha) = \sum \int_{\Delta^d} V(x)\,\mathrm{d}x.$$

One can now define the *filling volume* of a Lipschitz d-cycle α in X as

$$\mathrm{FV}_X^{d+1}(\alpha) = \inf_{\partial\beta=\alpha} \mathrm{mass}(\beta).$$

We then call some subset Z of X *undistorted up to dimension n* if there exists some constant $c > 0$ such that for any Lipschitz d-cycle α in Z with $d < n$, we have

$$\mathrm{FV}_Z^{d+1}(\alpha) \leq c\,\mathrm{FV}_X^{d+1}(\alpha) + c.$$

Young's proof of his theorem relies on a previous result by Urs Lang and Thilo Schlichenmaier [LS05]. In their proof they constructed a covering of a space $X \subseteq Z$ given that either X or Z has finite Assouad-Nagata-dimension. Young used this construction slightly modified and built with it a simplicial complex as the nerve of the cover. The simplicial complex is constructed in such a way that simplices near to Z have diameter $\lesssim \varepsilon$ where ε is free to choose. A well known result from geometric measure theory states that one can approximate Lipschitz chains with simplicial chains. Applying this so called deformation theorem is the main idea that is used in the proof of the main theorem.

[1]Note that if X is an arbitrary metric space, a generalized result holds for Lipschitz maps $\Delta^d \to X$ using metric differentiability. [Kir94]

Chapter 2

Preparations

> I write only for my shadow which is cast on the wall in front of the light. I must introduce myself to it.
>
> Sadegh Hedayat

2.1 Metric Spaces

Definition 1 Let X be a set and let $d : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a function such that for all $x, y, z \in X$

1. $d(x, y) > 0$ if $x \neq y$, and $d(x, x) = 0$, (Positiveness)
2. $d(x, y) = d(y, x)$, (Symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

Then d is called **metric**[1] and the space (X, d) is called **metric space**. [BI01]

Given a metric space (X, d) and a subset $Y \subseteq X$, define by $\operatorname{diam}(Y) := \sup \{d(x, y) \mid x, y \in Y\}$ the **diameter** of the set Y.[LS05]

Remark 1 Note that in a metric space (X, d) also the **reverse triangle inequality** holds for all $x, y, z \in X$

$$|d(x, y) - d(x, z)| \leq d(y, z). \tag{2.2}$$

[1] Sometimes also called **distance function** or simply **distance**.

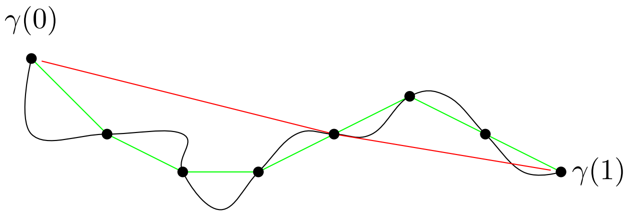

Figure 2.1: Definition of path length as supremum of sums over distances

2.1.1 Paths and Geodesics

Definition 2 Let (X, d) be a metric space. A continuous map $\gamma : [0,1] \to X$ is called a **path**. Define by

$$l(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(a_{i-1}), \gamma(a_i)) \,\middle|\, 0 = a_0 < a_1 < \ldots < a_n = 1 \right\}$$

the **length** of the path γ where the supremum is taken over all finite sequences $(a_0, a_1, \ldots, a_n)$ of the form $0 = a_0 < a_1 < \ldots < a_n = 1$ where $n < \infty$. A path γ with $l(\gamma) = d(\gamma(0), \gamma(1))$ is called a **geodesic path**. A metric space (X, d) in which for any two points there exists a geodesic path connecting them, is called a **geodesic metric space**. If for a metric space X there exists a $D > 0$ such that for any two points $x, y \in X$ there exists a path $\gamma : [0,1] \to X$ with endpoints x respectively y and length $l(\gamma) \leq D \cdot d(x, y)$, then the space (X, d) is called **quasi-convex** metric space. [Pap14]

2.2 Lipschitz Functions, Extensions and Homology

Integral to the entire following discussion is the notion of Lipschitz continuity which is defined as follows.

Definition 3 A function $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called **Lipschitz (continuous)** if there exists a constant $L \leq 0$ such that

$$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)$$

for any two points $x, y \in X$. The smallest possible choice for L is called **Lipschitz constant** and will be denoted by $\mathrm{Lip}(f)$. If f in addition has an inverse map $f^{-1} : f(X) \to X$ which is Lipschitz continuous as well, then f is called **bi-Lipschitz continuous**.

Notation 1 To emphasise the associated Lipschitz constant L of a function f, we may write f is L-Lipschitz continuous.

Furthermore one is interested in extension properties of Lipschitz functions.

Definition 4 Given two metric spaces as above together with a subset $X' \subseteq X$ as well as an L-Lipschitz function $f : X' \to Y$, if there exists a constant c and a Lipschitz function $\bar{f} : X \to Y$ such that

- $\bar{f}\Big|_{X'} = f$ and,
- $\mathrm{Lip}(\bar{f}) = cL$,

then $\bar{f}$ is called a **cL-Lipschitz extension** of f. If there exists a constant c' such that for any subset $X' \subseteq X$ and L-Lipschitz function $f : X' \to Y$ there exists a $c'L$-Lipschitz extension of f, then the pair $((X, d_X), (Y, d_Y))$ of metric spaces is said to have the **Lipschitz extension property**. [Sch05]

Notation 2 In this text the terms "map" and "function" are used interchangeably.

Notation 3 In the following the d-dimensional **closed unit disk** is denoted by

$$D^d := \left\{ \vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \,\Big|\, \|\vec{x}\| \leq 1 \right\},$$

and the d-dimensional **unit sphere** by

$$S^d := \left\{ \vec{x} = (x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \,\Big|\, \|\vec{x}\| = 1 \right\},$$

both are equipped with the induced metric.

Definition 5 A space X is called **Lipschitz n-connected** if there exists a constant $c > 0$ such that for any $0 \leq d \leq n$ and any L-Lipschitz map $f : S^d \to X$, there exists an cL-Lipschitz extension $\bar{f} : D^{d+1} \to X$ of f. [BF09; You14]

Definition 6 Let X be a space and $A \subseteq X$ a subset of X. A is called **Lipschitz retract** of X if there exists a Lipschitz function $f : X \to X$ such that

- $f\big|_A = \mathrm{id}_A$,
- $f(X) = A$.

A function f satisfying the above properties is called a **Lipschitz retraction**.[Hei05]

Notation 4 Let X be a set and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. We write $f \lesssim g$ if there exists a constant $c > 0$ such that $f(x) \leq cg(x)$ for all $x \in X$. Furthermore we write $f \sim g$ iff $f \lesssim g$ and $g \lesssim f$.

2.2.1 Homology

We continue by constructing a homology theory based on Lipschitz chains. This homology theory satisfies the Eilenberg–Steenrod axioms for a homology theory. See for example [RS09; DHP; Mon13].

Definition 7 (Homology theory based on Lipschitz chains) Let the following denote the **standard n-simplex** [Hat02]

$$\Delta^n := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_i t_i = 1 \text{ and } t_i \geq 0 \,\forall i \right\}.$$

A **singular n-simplex** in a given space X is a continuous map $\sigma : \Delta^n \to X$. If this map is additionally Lipschitz continuous then it is called **singular Lipschitz n-simplex**.

Given an abelian group G, finite formal sums of the form $\alpha = \sum_i n_i \sigma_i$ where $n_i \in G$ and $\sigma_i : \Delta^n \to X$ continuous are called **singular n-chains**. A **singular Lipschitz n-chain** is a singular n-chain $\alpha = \sum_i n_i \sigma_i$ where each $\sigma_i : \Delta^n \to X$ is Lipschitz continuous.

Denote by $\boldsymbol{C_n(X;G)}$ (respectively $\boldsymbol{C_n^{\mathrm{L}}(X;G)}$) the free abelian group with basis the singular n-simplices of X (respectively singular Lipschitz n-simplices of X) and coefficients in G. Furthermore write $C_n(X)$ for $C_n(X, \mathbb{Z})$ and $C_n^{\mathrm{L}}(X)$ for $C_n^{\mathrm{L}}(X, \mathbb{Z})$.

Furthermore given a simplicial complex X denote by $\boldsymbol{C_n^{\Delta}(X;G)}$ the free abelian group with basis the open n-simplices of X and coefficients in G. This consists of simplicial n-chains which can be written as finite formal sums $\sum_i n_i e_i^n$ where $n_i \in G$ and e_i^n is an open n-simplex of X. This is equivalent to a definition where we take the finite formal sums $\sum_i n_i \sigma_i$ where each $\sigma_i : \Delta^n \to X$ is the unique characteristic map of e_i^n with image the closure of e_i^n [Hat02]. As above we define $\boldsymbol{C_n^{\Delta}(X)} := \boldsymbol{C_n^{\Delta}(X;\mathbb{Z})}$.

Notation 5 In the following if not otherwise indicated maps are always continuous. If there is no risk of ambiguity we will write $C_n(X)$ instead of $C_n^{\Delta}(X)$. Furthermore from now on coefficients will always be in $\mathbb{Z}$.

Remark 2 Note that simplicial chains are by definition also Lipschitz chains.

The **boundary map** $\partial_{\mathrm{L}} : C_n(X) \to C_{n-1}(X)$ for Lipschitz n-chains can be defined [RS09] analogously to the case [Hat02] of singular n-chains. If the context is clear the boundary is written without the lower L simply as ∂.

Definition 8 Define the **boundary map** ∂ as follows. Given a Lipschitz n-chain $\alpha = \sum_i n_i \sigma_i$, let

$$\partial \alpha := \sum_i n_i \partial \sigma_i,$$

where we define for Lipschitz continuous $\sigma : \Delta^n \to X$ the boundary as

$$\partial \sigma := \sum_{i=0}^{n} (-1)^n \sigma \circ F_i^n.$$

Here $F_i^n : \Delta^{n-1} \to \Delta^n$ is the **i-th** face map defined as the map of the inclusion of the i-th face of Δ^n in Δ^n:

$$\Delta^{n-1} \cong \left\{ (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{k \neq i} t_k = 1 \text{ and } t_k \geq 0 \,\forall k \right\} \subseteq \Delta^n.$$

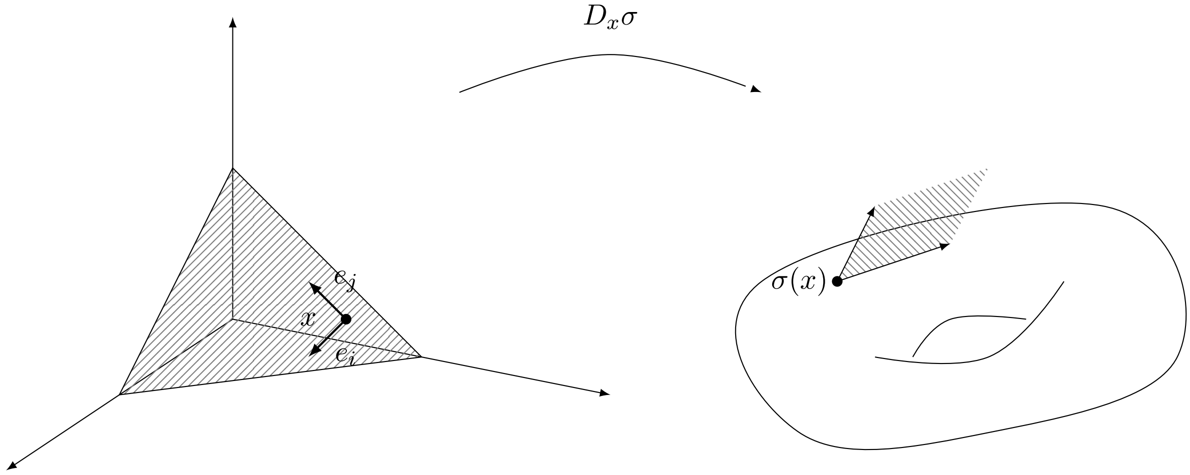

Figure 2.2: Parallelepiped used in the definition of mass

Definition 9 Elements of $Z_n(X) := \ker \partial_n$ are called **Lipschitz n-cycles** and elements of $B_n(X) := \operatorname{im} \partial_{n+1}$ are called **Lipschitz n-boundaries**. We may write $Z_n^{\Delta}(X), B_n^{\Delta}(X), Z_n^{\mathrm{L}}(X), B_n^{\mathrm{L}}(X)$ and so on to emphasise the associated chains and boundary maps.

Notation 6 Given two spaces respectively simplicial complexes X and Y together with their chain groups and a map $f : X \to Y$, we denote by $f_\sharp : C_\bullet(X) \to C_\bullet(Y)$ the canonical map on the level of chains.

2.2.2 Jacobian and Mass

Definition 10 ([Eps92]) Let X be a connected Riemannian manifold and let $\sigma : \Delta^k \to X$ be a singular Lipschitz k-simplex. By Rademacher's theorem (Theorem B.2.9), σ is differentiable almost everywhere. Therefore if the derivative $D_x\sigma$ at a given point $x \in \Delta^k$ exists it sends an orthonormal basis at x to a k-tuple of vectors tangent to X at $\sigma(x)$. Those vectors define a parallelepiped in the tangent space, with a volume $V(x) \geq 0$ given by the riemannian structure of X. The Lipschitz map may not be differentiable everywhere, in those cases we set $V(x) = 0$ for those points. See Figure 2.2.

Moreover define the **k-mass** (or simply **mass**) of a simplex as

$$\operatorname{mass}(\sigma) = \int_{\Delta^k} V(x)\,\mathrm{d}x$$

and the mass of a Lipschitz k-chain $\alpha = \sum_i n_i \sigma_i$ as

$$\operatorname{mass}(\alpha) = \sum_i |n_i| \operatorname{mass}(\sigma_i).$$

The mass may be written mass_k to emphasise the dimension.

Remark 3 Note that for an arbitrary metric space (X, d) a generalized version of Rademacher's theorem holds involving metric differentiability: Let $f : \mathbb{R}^n \to X$ be a Lipschitz continuous function, then the metric differential

$$\mathrm{MD}(f, p)(x) := \lim_{r \downarrow 0} \frac{d(f(p + rx), f(p))}{r}$$

of f exists in almost all points $p \in \mathbb{R}^n$ of $\mathbb{R}^n$. Furthermore one can assign a volume element $V(p)$ to each point $p \in \mathbb{R}^n$. Therefore the following discussion remains true if we consider arbitrary metric spaces instead of Riemannian manifolds.[Kir94; Bon14; AD09]

Proposition 1 *Let X and Y be connected Riemannian manifolds, $f : X \to Y$ an L-Lipschitz map and $\alpha \in C_k(X)$ a Lipschitz chain. Then:*

$$\mathrm{mass}(f_\sharp(\alpha)) \leq L^k \,\mathrm{mass}(\alpha).$$

Proof It suffices to show the statement for a singular Lipschitz k-simplex $\sigma : \Delta^k \to X$. Note that by the chain rule we have $D_x(f \circ \sigma) = D_{\sigma(x)} f \circ D_x\sigma$. Because f has Lipschitz constant L, for any basis vector $\vec{e_i}$ in an orthonormal basis at x we have the relation $\left|(D_{\sigma(x)} f \circ D_x\sigma)\vec{e_i}\right| \leq L \cdot |(D_x\sigma)\vec{e_i}|$. Adopting the notation $V_{f\circ\sigma}(x)$ and $V_\sigma(x)$ for the volume of the parallelepiped resulting from sending an orthonormal basis at x along $D_x(f \circ \sigma)$ respectively $D_x\sigma$ as defined above we see that

$$\begin{aligned} \mathrm{mass}(f \circ \sigma) &= \int_{\Delta^k} V_{f\circ\sigma}(x)\,\mathrm{d}x \\ &\leq \int_{\Delta^k} L^k \cdot V_\sigma(x)\,\mathrm{d}x = L^k \cdot \int_{\Delta^k} V_\sigma(x)\,\mathrm{d}x = L^k \,\mathrm{mass}(\sigma). \end{aligned}$$ □

2.3 Filling Volume and Assouad-Nagata Dimension

Definition 11 Define the **filling volume** [You08] of a Lipschitz n-cycle $\alpha \in C_n(X)$ as

$$\mathrm{FV}_X^{n+1}(\alpha) := \inf \left\{\mathrm{mass}(\alpha') \,\middle|\, \alpha' \in C_{n+1}(X) \text{ and } \partial\alpha' = \alpha\right\}.$$

Given $Z \subseteq X$, if there is some $c > 0$ such that for any Lipschitz n-cycle $\alpha \in Z_n(Z)$ with $n < m$ we have that

$$\mathrm{FV}_Z^{n+1}(\alpha) \leq c\,\mathrm{FV}_X^{n+1}(\alpha) + c$$

then we say that Z is **undistorted up to dimension** m [You14; BW07].

Corollary 1 *Given two connected Riemannian manifolds X and Y, a Lipschitz $(n-1)$-chain $\alpha \in C_{n-1}(X)$ and an L-Lipschitz function $f : X \to Y$. Then the filling volume satisfies the following relation:*

$$\mathrm{FV}_Y^n(f_\sharp(\alpha)) \leq L^n \,\mathrm{FV}_X^n(\alpha).$$

PROOF This directly follows from Proposition 1 as follows

$$\begin{aligned}\mathrm{FV}^n_Y(f_\sharp(\alpha)) &= \inf_{\partial\beta_Y = f_\sharp(\alpha)} \operatorname{mass}(\beta_Y)\\ &\le \inf_{\partial\beta_X=\alpha} \operatorname{mass}(f_\sharp(\beta_X))\\ &\le \inf_{\partial\beta_X=\alpha} L^n \operatorname{mass}(\beta_X) = L^n\,\mathrm{FV}^n_X(\alpha).\end{aligned}$$

Here $\beta_X \in C_n(X)$ and $\beta_Y \in C_n(Y)$ denote Lipschitz n-chains in X respectively Y. □

Notation 7 A covering by sets of diameter at most c is called a **c-bounded covering**. The **multiplicity** of a cover $\mathcal{B}$ is defined to be the maximal number of different sets in the cover with non-empty intersection.

$$\operatorname{multiplicity}(\mathcal{B}) := \max \# \{B_1, \dots, B_l \in \mathcal{B} \mid B_1 \cap \dots \cap B_l \neq \emptyset\}$$

Furthermore a cover is said to have **s-multiplicity** at most n, if any set D of diameter at most s intersects not more than n sets in the cover.

Remark 4 In some articles the multiplicity of a cover is given by the maximal number of different sets in the cover with non-empty intersection minus 1.

Definition 12 Define the **Assouad-Nagata dimension of** X as the smallest integer $n =: \dim_{\mathrm{AN}}(X)$ for which there exists a $c > 0$ such that for all $d > 0$ there is a cd-bounded covering $X = \bigcup_l X_l$ of X with d-multiplicity at most $n+1$.[You14]. Furthermore denote by $\mathrm{Const}_{\mathrm{AN}}(X) := c$ the implicit constant.

Proposition 2 (Proposition 2.2.1 in [Sch05]) *Given a space X, if $Y \subseteq X$ then $\dim_{AN}(Y) \le \dim_{AN}(X)$. Furthermore $\mathrm{Const}_{AN}(X) = \mathrm{Const}_{AN}(Y)$.*

2.4 Statement of the Main Theorem

We are now ready to state the main theorem

Theorem 1 (Theorem 1.3 in [You14]) *Let X be a quasi-convex metric space and let $Z \subset X$ be a non-empty closed subset with the metric given by the restriction of the metric of X and $\dim_{\mathrm{AN}}(Z) < \infty$.*[2] *Suppose that one of the following conditions is true:*

- *Z is Lipschitz n-connected or,*
- *X is Lipschitz n-connected, and if X_p, $p \in P$ are the connected components of $X \setminus Z$, then the sets $H_p = \partial X_p$ are Lipschitz n-connected with uniformly bounded implicit constant.*

Then Z is undistorted up to dimension $n+1$.

[2] Young stated the result for $\dim_{\mathrm{AN}}(X) < \infty$, we were able to improve the result and only require $\dim_{\mathrm{AN}}(Z) < \infty$.

The proof of the theorem is based on the proof of the following two theorems by Lang and Schlichenmaier:

Theorem 2 (Theorem 1.5 in [LS05]) *Let X and Y be metric spaces. Given a non-empty closed subset $Z \subset X$ with $\dim_{\mathrm{AN}}(X \setminus Z) \leq n < \infty$, if Y is Lipschitz $(n-1)$-connected, then there exists a constant $c \geq 0$ such that for any L-Lipschitz function $f : Z \to Y$ there exists a cL-Lipschitz extension $\bar{f} : X \to Y$ of f.*

Using the following second version Lang and Schlichenmaier gave of the theorem, we can improve the conditions of the theorem, thus only requiring $\dim_{\mathrm{AN}}(Z) < \infty$ instead of $\dim_{\mathrm{AN}}(X) < \infty$.

Theorem 3 (Theorem 1.6 in [LS05]) *Let X and Y be metric spaces. Given a non-empty closed subset $Z \subset X$ with $\dim_{\mathrm{AN}}(Z) \leq n - 1 < \infty$, if Y is Lipschitz $(n-1)$-connected, then there exists a constant $c \geq 0$ such that for any L-Lipschitz function $f : Z \to Y$ there exists a cL-Lipschitz extension $\bar{f} : X \to Y$ of f.*

Two small corollaries follow directly from Lang and Schlichenmaier's two theorems.

Corollary 2 ([You14]) *Let X be a metric space and $Z \subset X$ a Lipschitz $(n-1)$-connected, non-empty closed subset with $\dim_{\mathrm{AN}}(X \setminus Z) \leq n < \infty$. Then Z is a Lipschitz retract of X and Z is undistorted up to any dimension.*

Proof Consider the identity map $i : Z \to Z$. By Theorem 2.4.2 this can be extended to a cL-Lipschitz map $\bar{i} : X \to Z$. This map is a retraction of X onto Z. Let $\alpha \in Z_{k-1}(Z)$, $\beta \in C_k(X)$ such that $\partial\beta = \alpha$, then:

$$\partial \bar{i}_\sharp(\beta) = \bar{i}_\sharp(\partial\beta) = \bar{i}_\sharp(\alpha) = \alpha$$

and

$$\bar{i}_\sharp(\beta) \in C_k(Z).$$

Using Corollary 1 we get that $\mathrm{FV}^k_Z(\alpha) \leq (cL)^k\, \mathrm{FV}^k_X(\alpha)$ which proves the statement. □

Corollary 3 ([You14]) *Let X be a metric space and $Z \subset X$ a Lipschitz $(n-1)$-connected, non-empty closed subset with $\dim_{\mathrm{AN}}(Z) \leq n - 1 < \infty$. Then Z is a Lipschitz retract of X and Z is undistorted up to any dimension.*

Proof The same proof as in the previous corollary works here as well. □

The proof will require further definitions and tools which we will introduce in the following sections.

2.5 Riemannian Simplical Complexes and QC Complexes

Definition 13 A **Riemannian simplicial complex** is a simplicial complex X together with a metric d_X which gives each simplex the structure of a Riemannian manifold with corners.[You14; Joy09] If additionally there exists a constant C such that for all simplices $\Delta^k \in X$ and all $x, y \in \Delta^k$ we have that $C^{-1} d_\Delta(x, y) \leq d_X(x, y) \leq C d_\Delta(x, y)$, where d_Δ is the metric on some scaling of the standard k-simplex, then we call X a **QC complex** (or quasi-conformal complex).[You14]

Two variations of the following well known theorem from geometric measure theory will be used in the constructions for the proof.

Theorem 4 (Deformation Theorem (Theorem 10.3.3 in [Eps92])) *Let M be a Riemannian manifold and $(\tau, h : \tau \to M)$ be a triangulation (that is τ is a simplicial complex homeomorphic to M with h the associated homeomorphism) of M. Then there exists a constant $c > 0$ such that for each Lipschitz k-cycle $\alpha \in Z_k^L(M)$ there exist a smooth cycle $\tilde{\alpha} \in Z_k^\Delta(\tau)$ whose simplices all consist of simplicial maps and a chain $\beta \in C_{k+1}^L(M)$ such that:*

- $\alpha = \tilde{\alpha} + \partial\beta$,
- $\mathrm{mass}_k(\tilde{\alpha}) \leq c \cdot \mathrm{mass}_k(\alpha)$,
- $\mathrm{mass}_{k+1}(\beta) \leq c \cdot \mathrm{mass}_k(\tilde{\alpha})$.

Furthermore $\tilde{\alpha}$ and β are contained in the smallest subcomplex of τ containing α.

For simplicial complexes we have the following two versions. They will be used during the proof of the main theorem and in the preceding lemmas.

Theorem 5 (Deformation Theorem for simplicial complexes (Theorem 2.2 in [You14])) *Let Σ be a simplical complex where each simplex has the metric of the standard simplex of diameter s (a scaled simplicial complex). Then there is a constant c depending only on $\dim(\Sigma)$ such that for all Lipschitz k-chains[3] $\alpha \in C_k^L(\Sigma)$ with $\partial\alpha \in C_{k-1}(\Sigma)$ simplicial there exist a simplicial k-chain $P(\alpha) \in C_k(\Sigma)$ and a Lipschitz $(k+1)$-chain $Q(\alpha) \in C_{k+1}^L(\Sigma)$ such that:*

1. $\mathrm{mass}(P(\alpha)) \leq c \cdot \mathrm{mass}(\alpha)$,
2. $\mathrm{mass}(Q(\alpha)) \leq cs \cdot \mathrm{mass}(\alpha)$,
3. $\partial Q(\alpha) = \alpha - P(\alpha)$,
4. $\partial\alpha = \partial P(\alpha)$.

[3] Young requires α to be a cycle, but in the applications of the theorem in the article α is not a cycle. We will instead require $\partial\alpha$ to be simplicial.

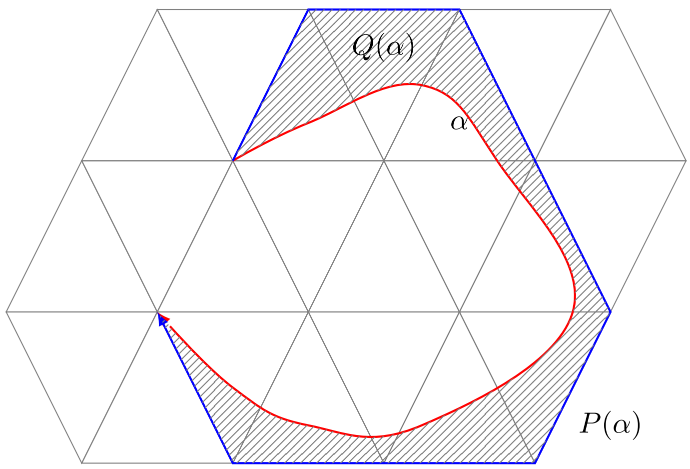

Figure 2.3: Illustration of the deformation theorem for simplicial complexes

Corollary 4 (Theorem 2.3 in [You14]) *Let Σ be a QC complex. Then there exists a constant $c > 0$ depending only on $\dim \Sigma$ such that for all Lipschitz k-chains $\alpha \in C_k^L(\Sigma)$ with $\partial\alpha \in C_{k-1}(\Sigma)$ simplicial there exists a simplicial k-chain $P(\alpha) \in C_k(\Sigma)$ approximating α such that $\operatorname{mass} P(\alpha) \leq c \cdot \operatorname{mass}(\alpha)$ and $\partial P(\alpha) = \partial\alpha$.*

The deformation theorem for simplicial complexes is illustrated in Figure 2.3.

Part II

Proof of the Main Theorem

Chapter 3

Proof of the Main Theorem

If you have to dry the dishes
(Such an awful boring chore)
If you have to dry the dishes
('Stead of going to the store)
If you have to dry the dishes
And you drop one on the floor
Maybe they won't let you
Dry the dishes anymore

Shel Silverstein

This chapter will consist of the proof of the main theorem (Theorem 2.4.1).

3.1 Outline of the Proof of the Main Theorem

1. Lemma 3: Let $\varepsilon > 0$. Cover X by sets such that sets close to Z have small diameter ($\lesssim \varepsilon$) whereas sets far away have large diameter and the covering of Z is $4\varepsilon(c+1)$-bounded.
2. Construct a Riemannian simplicial complex Σ as the nerve of the cover from the previous step. Endow it with a metric such that the diameter of each simplex is comparable to the diameter of the sets that give it its vertices. This makes Σ a QC complex.
3. Lemma 4: Construct a Lipschitz function $g : X \to \Sigma$ with Lipschitz constant independent of ε sending points in X to the QC complex in a natural way.
4. Lemma 6: Show that the second condition in the main theorem implies the first one. We therefore only need to show the theorem in case condition one holds.
5. Lemma 7: Construct a Lipschitz function $h : \Sigma^{(0)} \to Z$ with Lipschitz constant independent of ε and such that $d(h \circ g(z), z) \lesssim \varepsilon$ for every $z \in Z$.

6. Lemma 9: Construct a simplicial chain α' and two annuli γ and λ as approximations like shown in Figure 3.5. Estimate the masses of the annuli.
7. Proof on page 34: Given a Lipschitz cycle α in Z and a Lipschitz cycle β in X, use the deformation theorem to approximate $g_\sharp(\beta) - \gamma$ by a simplicial cycle P in Σ. This in turn gives us a Lipschitz chain $\lambda + h_\sharp(P)$ in Z. We will find that $\partial(\lambda + h_\sharp(P)) = \alpha$. And furthermore that the mass of $\lambda + h_\sharp(P)$ is comparable to the mass of β.

3.2 Proof of the Main Theorem

In the proof of Theorem 1.5 in the article of Lang and Schlichenmaier [LS05] they constructed a covering. This result has been condensed into the following lemma. It will be used in constructing another cover in the Lemma 3 below.

Lemma 1 (Proof of Theorem 1.5 / Application of Theorem 5.2 in [LS05]) *Let X be a metric space. Furthermore let $Z \subset X$ be a non-empty, closed subset of X with $\dim_{\mathrm{AN}}(X \setminus Z) < \infty$. Then there exist constants $\alpha, \beta > 0$ and a covering $\mathcal{B} = \{B_i\}_{i \in I}$ of $X \setminus Z$ where $B_i \subseteq X \setminus Z$, such that:*

1. $\operatorname{diam}(B_i) \leq \alpha d(B_i, Z)$ *for all $i \in I$,*
2. *every set $D \subseteq X \setminus Z$ with* $\operatorname{diam}(D) \leq \beta d(D, Z)$ *meets at most* $\dim_{AN}(X \setminus Z) + 1$ *members of $\mathcal{B}$.*

Furthermore the function $\sigma_i : X \setminus Z \to \mathbb{R}$, given by

$$\sigma_i = \max\{0, \delta d(B_i, Z) - d(x, B_i)\}$$

with $\delta := \frac{\beta}{2(\beta+1)}$ satisfies

$$\#\{i \in I \mid \sigma_i(x) > 0\} \leq \dim_{AN}(X \setminus Z) + 1.$$

Remark 5 Explicitly the constants needed for the above lemma are given in the proof of the theorem as: $\alpha = 2c + 1$, $\beta = \frac{1}{3+2c}$, where $c := \mathrm{Const}_{\mathrm{AN}}(X \setminus Z)$. It follows that $\alpha > 1$ and $0 < \beta < \frac{1}{3}$ and furthermore $\alpha = \beta^{-1} - 2$.

A similar version of the previous lemma exists which only requires the Assouad-Nagata dimension of Z to be finite.

Lemma 2 (Proof of Theorem 1.6 / Application of Theorem 5.2 in [LS05]) *Let X be a metric space. Furthermore let $Z \subset X$ be a non-empty, closed subset of X with $\dim_{\mathrm{AN}}(Z) < \infty$. Then there exist constants $\alpha, \beta > 0$ and a covering $\mathcal{B} = \{B_i\}_{i \in I}$ of $X \setminus Z$ where $B_i \subseteq X \setminus Z$, such that:*

1. $\mathrm{diam}(B_i) \leq \alpha d(B_i, Z)$ *for all* $i \in I$,
2. *every set* $D \subseteq X \setminus Z$ *with* $\mathrm{diam}(D) \leq \beta d(D, Z)$ *meets at most* $\dim_{AN}(Z) + 2$ *members of* $\mathcal{B}$.

Furthermore the function $\sigma_i : X \setminus Z \to \mathbb{R}$, *given by*

$$\sigma_i = \max\{0, \delta d(B_i, Z) - d(x, B_i)\}$$

with $\delta := \frac{\beta}{2(\beta+1)}$ *satisfies*

$$\#\{i \in I \mid \sigma_i(x) > 0\} \leq \dim_{AN}(Z) + 2.$$

Remark 6 Explicitly the constants needed for the above lemma are given in the proof of the theorem as: $\alpha = 56 + 138c' + 108c'^2 + 27c'^3$, $\beta = \min\{1, \sqrt[n]{4 + 3c'} - 1\}$, where $c := \mathrm{Const}_{\mathrm{AN}}(Z)$ and $c' \geq 0$ depends only on c and n. It follows that $\alpha > 1$ and $0 < \beta < 1$.

We can make sure that each set B_i of the covering is contained in a connected component of $X \setminus Z$ by slightly weakening the second property of the lemma:

Corollary 5 *Let* X *be a metric space. Furthermore let* $Z \subset X$ *be a non-empty, closed subset of* X *with* $\dim_{\mathrm{AN}}(Z) < \infty$. *Then there exist constants* $\alpha, \beta > 0$ *and a covering* $\mathcal{B} = \{B_i\}_{i \in I}$ *of* $X \setminus Z$ *where* $B_i \subseteq X \setminus Z$ *and each* B_i *is contained in a connected component of* $X \setminus Z$, *such that:*

1. $\mathrm{diam}(B_i) \leq \alpha d(B_i, Z)$ *for all* $i \in I$,
2. *every set* $D \subseteq X \setminus Z$ *with* $\mathrm{diam}(D) \leq \beta d(D, Z)$ *which is contained in a connected component of* $X \setminus Z$ *meets at most* $\dim_{AN}(Z) + 2$ *members of* $\mathcal{B}$.

Furthermore the function $\sigma_i : X \setminus Z \to \mathbb{R}$, *given by*

$$\sigma_i = \max\{0, \delta d(B_i, Z) - d(x, B_i)\}$$

with $\delta := \frac{\beta}{2(\beta+1)}$ *satisfies*

$$\#\{i \in I \mid \sigma_i(x) > 0\} \leq \dim_{AN}(Z) + 2.$$

Proof Let $\bar{\mathcal{B}} = \{B_i\}_{i \in I}$ be the cover from the lemma above. If X_p, $p \in P$ are the connected components of $X \setminus Z$, then let $B_i^p := B_i \cap X_p$ and let $\mathcal{B} := \{B_i^p \mid i \in I, p \in P\}$. These sets satisfy the following properties:

- $d(B_i, Z) \leq d(B_i^p, Z)$ for any $i \in I$ and $p \in P$,
- $\mathrm{diam}(B_i^p) \leq \mathrm{diam}(B_i)$ for any $i \in I$.

We check the required properties:

- $\operatorname{diam}(B_i^p) \le \operatorname{diam}(B_i) \le \alpha d(B_i, Z) \le \alpha d(B_i^p, Z)$,
- on each connected component of $X \setminus Z$, the number of sets covering it remains the same, because for any $p \in P$

$$\#\{i \in I \mid B_i \cap X_p \neq \emptyset\} = \#\{(i,q) \in I \times P \mid B_i \cap X_p \neq \emptyset, q = p\}. \qquad \square$$

Lemma 3 (Lemma 2.4 in [You14]) *Let X be a metric space and $Z \subset X$ a non-empty, closed subset of X with $\dim_{AN}(Z) < \infty$. Then there are constants $\alpha, \beta, \gamma > 0$ and $\delta := \frac{\beta}{2(\beta+1)}$ depending only on X and Z, such that for each $\varepsilon > 0$ there exists a covering $\mathcal{D} = \{D_k\}_{k\in K} = \left\{\hat{B}_{\hat{\imath}}\right\}_{\hat{\imath}\in\hat{I}} \cup \left\{\bar{C}_j\right\}_{j\in J}$ of X where $D_k \subseteq X$ and functions $r : K \to \mathbb{R}$, $\tau_k : X \to \mathbb{R}$ given by*

$$r(k) = \begin{cases} \delta \cdot d(D_k, Z), & k \in \hat{I} \\ \varepsilon, & k \in J \end{cases}$$

$$\tau_k(x) = \max\{0,\, r(k) - d(x, D_k)\}$$

such that for all $k \in K$:

1. $\operatorname{diam}(D_k) \lesssim r(k)$,
2. $d(D_k, Z) \lesssim r(k)$,
3. *If $d(D_k, Z) \ge \varepsilon$, then $\{\tau_k > 0\}$ is contained in a connected component of $X \setminus Z$,*
4. *The cover of X given by $\{\{\tau_k > 0\}\}_{k\in K}$ has multiplicity at most $2\dim_{AN}(Z) + 3$,*
5. *If $\{\tau_k > 0\} \cap \{\tau_{k'} > 0\} \neq \emptyset$, then $\gamma^{-1} r(k') \le r(k) \le \gamma\, r(k')$.*

Proof We start by constructing the cover $\mathcal{D}$ using the cover from the Corollary 5 to the Lemma 2 above. This guarantees that each $B_i \in \mathcal{B}$ is contained in a connected component of $X \setminus Z$. Let $N_\varepsilon(Z) := \{x \in X \mid d(x, Z) < \varepsilon\}$ denote the open ε-neighbourhood of Z. Furthermore let

$$\hat{I} := \{\hat{\imath} \in I \mid B_{\hat{\imath}} \not\subseteq N_\varepsilon(Z)\}$$

and set $\hat{B}_{\hat{\imath}} := B_{\hat{\imath}} \cap (X \setminus N_\varepsilon(Z))$. Denote this cover by $\hat{\mathcal{B}} := \left\{\hat{B}_{\hat{\imath}}\right\}_{\hat{\imath}\in\hat{I}}$. $\hat{\mathcal{B}}$ covers $X \setminus N_\varepsilon(Z)$.

Let $c := \mathrm{Const}_{\mathrm{AN}}(Z)$. Since $\dim_{\mathrm{AN}}(Z) < \infty$, for $s := 4\varepsilon$ there exists a cs-bounded covering $\mathcal{C} := \{C_j\}_{j\in J}$ of Z with sets $C_j \subseteq Z$ in Z and with s-multiplicity at most $\dim_{\mathrm{AN}}(Z) + 1$. We increase the thickness of those sets by setting $\bar{C}_j := N_\varepsilon(C_j) \subseteq N_\varepsilon(Z)$. For the resulting sets we know that

$$\operatorname{diam}(\bar{C}_j) \le 4\varepsilon c + 2\varepsilon.$$

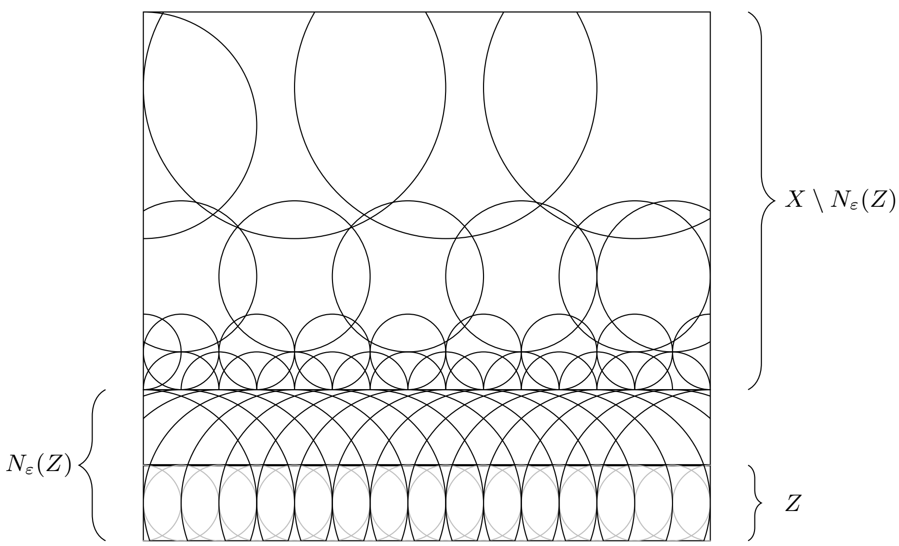

Figure 3.1: Illustration of the cover constructed in Lemma 3.

We denote this covering by $\bar{\mathcal{C}} := \left\{\bar{C}_j\right\}_{j\in J}$. Let $\mathcal{D} := \hat{\mathcal{B}} \cup \bar{\mathcal{C}}$ and furthermore $K := \hat{I} \sqcup J$. We will call sets D_k with $k \in \hat{I}$ of kind B and if $k \in J$ then of kind C. The constructed cover is illustrated in Figure 3.1.

For sets D_k of kind B, we know that because D_k may not lie in $N_\varepsilon(Z)$ therefore the following inequality must hold

$$\varepsilon \leq d(D_k, Z). \tag{3.2}$$

For (1), note that if D_k is of kind B then $\operatorname{diam}(D_k) \leq \alpha d(D_k, Z)$ by item 1 in Corollary 5 above. If D_k is of kind C then $\operatorname{diam}(D_k) \leq 4c\varepsilon + 2\varepsilon = \varepsilon(4c+2)$. We can therefore take $c_1 := \max\left\{\alpha\delta^{-1}, (4c+2)\right\}$ as constant and we have that $\operatorname{diam}(D_k) \leq c_1 r(k)$.

To show (2), consider that if D_k is of kind B then $r(k) = \delta d(D_k, Z)$. Clearly $d(D_k, Z) = \delta^{-1} r(k)$. Otherwise if D_k is of kind C then $r(k) = \varepsilon$ and $\delta d(D_k, Z) = 0$. Thus we can take the constant to be $c_2 := \delta^{-1}$ and get $d(D_K, Z) \leq \delta^{-1} r(k)$.

Statement (3) can be shown as follows. If D_k is such that $d(D_k, Z) \geq \varepsilon$ then D_k is clearly of kind B and $r(k) = \delta d(D_k, Z)$. From the definition of τ we can see that $\tau_k\big|_{D_k} \equiv r(k)$, so $\{\tau_k > 0\}$ lies in the same connected component as D_k.

For (4), consider the following. We will first show that the cover $\{\{\tau_k > 0\}\}_{k\in\hat{I}}$ has multiplicity at most $\dim_{\mathrm{AN}}(Z) + 2$. To do this we can follow the same argument as in the proof of the properties of σ in [LS05]: Let $x \in X \setminus Z$. For every $k \in \hat{I}$ with $x \in \{\tau_k > 0\}$ take an $x_k \in D_k$ with $d(x, x_k) < r(k) = \delta d(D_k, Z)$. We then know that $r(k) = \delta d(D_k, Z) \leq \delta d(x_k, Z)$. Let D be the set of those x_k. The set D satisfies

$$\operatorname{diam}(D) = \sup\left\{d(y, y') \mid y, y' \in D\right\} \leq 2\delta \sup_k d(x_k, Z) \leq 2\delta(\operatorname{diam}(D) + d(D, Z)).$$

Substituting the constants and calculating

$$\operatorname{diam}(D) \leq 2\frac{\beta}{2(\beta+1)}(\operatorname{diam}(D) + d(D, Z)) \iff \operatorname{diam}(D) \leq \beta d(D, Z)$$

shows that D satisfies condition 2 of Corollary 5. So the cover $\{\{\tau_k > 0\}\}_{k\in\hat{I}}$ has multiplicity at most $\dim_{\mathrm{AN}}(Z) + 2$.

It remains to show that the cover $\{\{\tau_k > 0\}\}_{k\in J}$ has multiplicity at most $\dim_{\mathrm{AN}}(Z) + 1$. Assume for a contradiction, that there is a $x \in X$ such that

$$\sharp \{j \in J \mid \{\tau_j > 0\} \cap \{x\} \neq \emptyset\} > \dim_{\mathrm{AN}}(Z) + 1.$$

This is equivalent to the condition

$$\sharp \left\{ j \in J \,\middle|\, \bar{C}_j \cap N_\varepsilon(\{x\}) \neq \emptyset \right\} > \dim_{\mathrm{AN}}(Z) + 1,$$

which we can further rewrite as

$$\sharp \{j \in J \mid C_j \cap N_{2\varepsilon}(\{x\}) \neq \emptyset\} > \dim_{\mathrm{AN}}(Z) + 1.$$

But we know that $\operatorname{diam}(N_{2\varepsilon}(\{x\})) \leq 4\varepsilon$ and therefore $N_{2\varepsilon}(\{x\})$ intersects at most $\dim_{\mathrm{AN}}(Z) + 1$ sets in the cover $\{C_j\}_{j\in J}$, which is a contradiction.

We have thus shown that the cover of X by sets $\{\{\tau_k > 0\}\}_{k\in K}$ has multiplicity at most $2\dim_{\mathrm{AN}}(X) + 3$.

For point (5) assume $\{\tau_k > 0\} \cap \{\tau_{k'} > 0\} \neq \emptyset$ and consider the two cases $r(k') = \varepsilon$ and $r(k') = \delta d(D'_k, Z)$:

1. Suppose $r(k') = \varepsilon$, then either $r(k) = \varepsilon$ or $\varepsilon \leq d(D_k, Z) = \delta^{-1} r(k)$ by (3.2).
2. If on the other hand $r(k') = \delta d(D_{k'}, Z)$, then one can build a path with lengths multiples of $r(k)$ from $D_{k'}$ to Z, see Figure 3.2. From the figure we can see that

$$\delta^{-1} r(k') < \delta^{-1} r(k) + r(k)c_1 + r(k) + r(k')$$

which we can rewrite as

$$r(k')(\delta^{-1} - 1) < r(k)(\delta^{-1} + 1 + c_1).$$

Because of $0 < \delta < \frac{1}{2}$ we know that $\delta^{-1} > 2$ and we can divide to get

$$r(k') < r(k)\frac{\delta^{-1} + 1 + c_1}{\delta^{-1} - 1}.$$

Finally by symmetry we get that

$$\frac{\delta^{-1} - 1}{\delta^{-1} + 1 + c_1} r(k) < r(k') < r(k)\frac{\delta^{-1} + 1 + c_1}{\delta^{-1} - 1}$$

which is what we needed to show. □

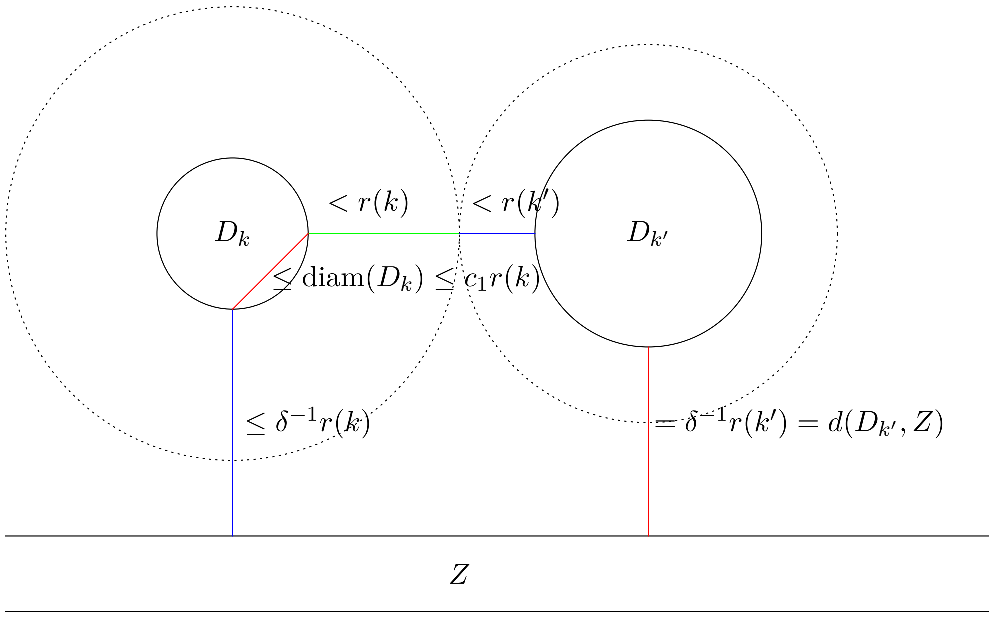

Figure 3.2: Construction for proof of Lemma 3, item 5.

Corollary 6 *The function* $\tau_k(\cdot)$ *is* 1*-Lipschitz.*

Proof Let $x, y \in X$. If $\tau_k(x) \neq 0$ and $\tau_k(y) \neq 0$ then

$$|\tau_k(x) - \tau_k(y)| = |d(x, D_k) - d(y, D_k)| \leq d(x, y)$$

by the reverse triangle inequality (2.2). If on the other hand $\tau_k(y) = 0$ then $d(y, D_k) \geq r(k)$ and therefore

$$|\tau_k(x)| \leq |r(k) - d(x, D_k)| \leq |d(y, D_k) - d(x, D_k)| \leq d(x, y). \qquad \square$$

Definition 14 Given a covering $\mathcal{C}$ of a space X, define the **nerve of the cover** [Bot95] $\mathcal{C}$ to be the simplicial complex $\mathcal{K}$ constructed as follows:

- There is one vertex v_i for each element $U_i \in \mathcal{C}$,
- There is an edge connecting v_i and v_j iff $U_i \cap U_j \neq \emptyset$ for the corresponding sets in the cover,
- And in general, there is a n-simplex for each $n+1$ element subset $\mathcal{U} \subseteq \mathcal{C}$ for which $\bigcap \mathcal{U} \neq \emptyset$.

Let Σ be the nerve of the cover $\{\{\tau_k > 0\}\}_{k \in K}$ constructed in the lemma above. Denote the vertex set by $\mathcal{V}(\Sigma) = \{v_k\}_{k \in K}$. Let $s : \Sigma \to \mathbb{R}$ be a function such that $s(v_k) = r(k)$ on each vertex and s is affine on each simplex. Furthermore define a Riemannian metric x_c on each simplex of Σ by $\mathrm{d}x_c^2 = s^2\,\mathrm{d}x^2$. Denote the distance function induced by this metric by $d_C : \Sigma \times \Sigma \to \mathbb{R}_{\geq 0}$.

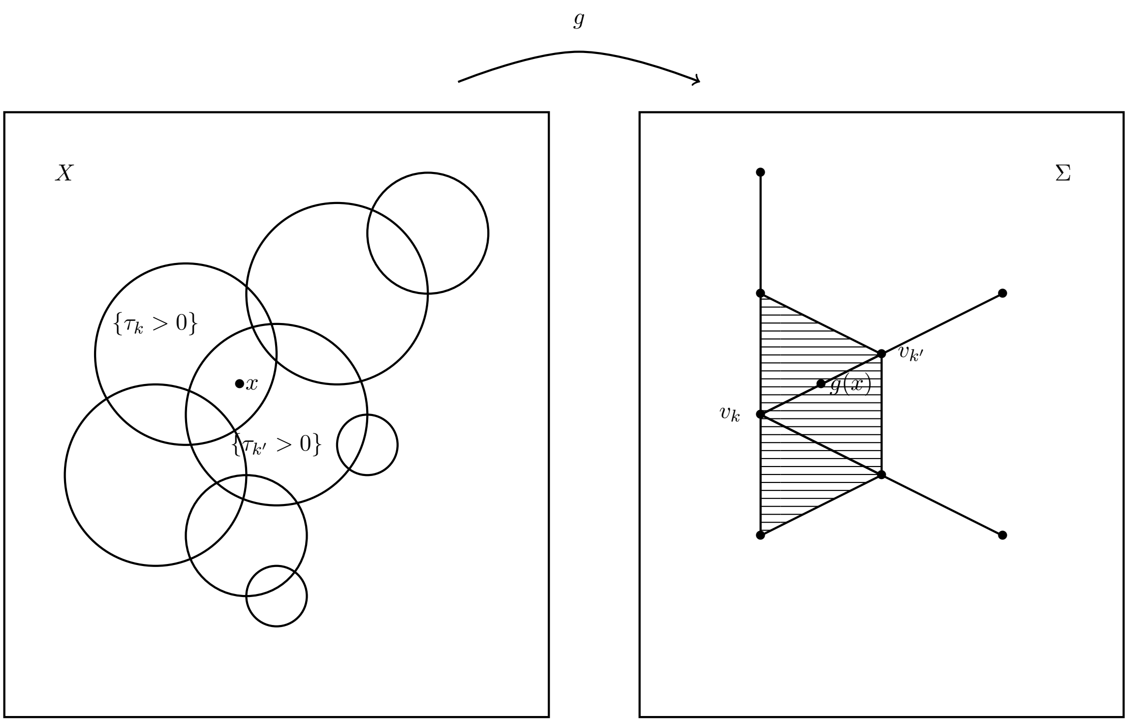

Figure 3.3: Figure for Lemma 4

Corollary 7 *Given a simplex $\sigma = \langle v_{k_1}, \ldots, v_{k_n} \rangle$ of Σ, then the function s satisfies $\gamma^{-1} r(k_1) \leq s \leq \gamma r(k_1)$ on σ. Therefore the metric makes Σ into a quasi-conformal complex.*

Definition 15 Given a simplicial complex $\mathcal{K}$ and a vertex v in $\mathcal{K}$, **the star of** v is defined as the union of all simplices in $\mathcal{C}$ having v as a vertex. [Bot95]

Lemma 4 (Lemma 2.5 in [You14]) *Under the assumptions of the main theorem, there exists a Lipschitz map $g : X \to \Sigma$ with $\operatorname{Lip}(g)$ independent of ε. Furthermore, if $x \in \{\tau_k > 0\}$, then $g(x)$ is in the star of v_k.*

Proof Remember the definition of the standard simplex

$$\Delta^n := \left\{ \vec{v} = (v_0, \ldots, v_n) \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \|\vec{v}\|_1 := \sum_{i=0}^{n} v_i = 1 \right\}$$

and note that this can also be viewed as[1]

$$\Delta^n_K := \left\{ p : (n+1) \to [0,1] \;\middle|\; \|p\|_1 := \sum_{i=0}^{n} p(i) = 1 \right\}.$$

[1]Here $(n+1)$ denotes the discrete set of $n+1$ elements.

Define by

$$\Delta_K := \{p : K \to [0,1] \mid \|p\|_1 = 1\}$$

the **infinite simplex** where K is the set from the construction above. We continue by constructing an injective simplicial map $\Sigma \hookrightarrow \Delta_K$. Because simplicial maps are determined by the effect of the map on the vertices it suffices to send v_k to $p_k : K \to [0,1]$ where

$$p_k(i) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}.$$

Using this injection we may view Σ as a subcomplex of Δ_K.

Let

$$g(x)(k) := \frac{\tau_k(x)}{\bar{\tau}(x)}$$

where

$$\bar{\tau}(x) = \sum_{k \in K} \tau_k(x).$$

Using a compact notation we may write $g_k(x)$ for $g(x)(k)$. Note that $g(x) : K \to [0,1]$, and thus g defines a function $g : X \to \Sigma \subseteq \Delta_K$. We need to show that g is Lipschitz. Because X is a quasi-convex metric space it is sufficient to show the condition only for points "close" to each other. Let $x, y \in X$ be such that $d(x,y) < \delta^2\varepsilon$. Let S_x and S_y be the smallest simplices of Σ which contain $g(x)$ respectively $g(y)$.

<u>*Claim 1:*</u> S_x and S_y share one common vertex v_m.

Proof of Claim 1: Consider the following cases:

1. If $d(x,Z) < \varepsilon$ then, there exists a D_k of kind C with $x \in D_k$ and from $r(k) = \varepsilon$ it follows $\tau_k(x) = \max\{0, \varepsilon - d(x, D_k)\} = \varepsilon$. Because τ_k is 1-Lipschitz we know that $|\varepsilon - \tau_k(y)| \leq d(x,y) < \delta^2\varepsilon$ which implies $\tau_k(y) \geq \varepsilon(1-\delta^2) > 0$ and we can take $m = k$.
2. If $d(x,Z) \geq \varepsilon$ then there is some D_k of kind B with $x \in D_k$ and $r(k) = \delta d(D_k, Z)$ therefore $\tau_k(x) = \delta d(D_k, Z)$. From equation (3.2) we know $\delta d(D_k, Z) \geq \varepsilon$ so $\tau_k(x) \geq \varepsilon$ which forces $\tau_k(y) > 0$. We can take $m = k$. ■

<u>*Claim 2:*</u> There exists an $L > 0$ such that $d_C(g(x), g(y)) \leq L \cdot d(x,y)$ for all $x, y \in X$ with $d(x,y) < \delta^2\varepsilon$.

Proof of Claim 2: From Lemma 3 and Corollary 7 we know that $\gamma^{-1}\, r(m) \leq s \leq \gamma\, r(m)$ on $S_x \cup S_y$. The Riemannian metric on Σ induces a distance function d_C. If g^m denotes the metric tensor, we can write

$$d_C(g(x), g(y)) = \inf_{\gamma_p} \int_a^b \sqrt{g^m_{\gamma_p(t)}(\dot{\gamma_p}(t), \dot{\gamma_p}(t))}\, \mathrm{d}t,$$

where the infinum runs over all paths $\gamma_p : [a,b] \to \Sigma$, with $a < b$ and $\gamma_p(a) = g(x)$ and $\gamma_p(b) = g(y)$. We calculate[2]

$$\begin{aligned}
d_C(g(x), g(y)) &= \inf_{\gamma_p} \int_0^1 \sqrt{s(\gamma_p(t))^2 \cdot \dot{\gamma_p}(t)^2}\,\mathrm{d}t \\
&\leq \int_0^1 \sqrt{s(g(x)(1-t) + g(y)t)^2 \cdot \sum_{k\in\Sigma^{(0)}} |g(x)(k) - g(y)(k)|^2}\,\mathrm{d}t \\
&\leq \int_0^1 \sqrt{(\gamma r(m))^2 \cdot \sum_{k\in\Sigma^{(0)}} |g(x)(k) - g(y)(k)|^2}\,\mathrm{d}t \\
&= \int_0^1 (\gamma r(m)) \cdot \sqrt{\sum_{k\in\Sigma^{(0)}} |g(x)(k) - g(y)(k)|^2}\,\mathrm{d}t \\
&\leq \sum_{k\in\Sigma^{(0)}} \int_0^1 (\gamma r(m)) \cdot |g(x)(k) - g(y)(k)|\,\mathrm{d}t \\
&= \sum_{k\in\Sigma^{(0)}} (\gamma r(m)) \cdot |g(x)(k) - g(y)(k)| \\
&= \gamma r(m) \sum_{k\in\Sigma^{(0)}} |g(x)(k) - g(y)(k)| \\
&= \gamma r(m) \sum_{k\in(S_x\cup S_y)^{(0)}} |g(x)(k) - g(y)(k)| \\
&= \gamma r(m) \sum_{k\in(S_x\cup S_y)^{(0)}} \left| \frac{\tau_k(x)}{\bar{\tau}(x)} - \frac{\tau_k(y)}{\bar{\tau}(y)} \right| \\
&= \gamma r(m) \sum_{k\in(S_x\cup S_y)^{(0)}} \left| \frac{\tau_k(x)}{\bar{\tau}(x)} - \frac{\tau_k(y)}{\bar{\tau}(y)} + \frac{\tau_k(y)}{\bar{\tau}(x)} - \frac{\tau_k(y)}{\bar{\tau}(x)} \right| \\
&\leq \gamma r(m) \sum_{k\in(S_x\cup S_y)^{(0)}} \left| \frac{\tau_k(x)}{\bar{\tau}(x)} - \frac{\tau_k(y)}{\bar{\tau}(x)} \right| + \left| \frac{\tau_k(y)}{\bar{\tau}(x)} - \frac{\tau_k(y)}{\bar{\tau}(y)} \right| \\
&\gamma r(m) \sum_{k\in(S_x\cup S_y)^{(0)}} \frac{1}{\bar{\tau}(x)} \left(|\tau_k(x) - \tau_k(y)| + \frac{\tau_k(y)}{\bar{\tau}(y)} |\bar{\tau}(x) - \bar{\tau}(y)| \right) \\
&= \frac{\gamma r(m)}{\bar{\tau}(x)} \sum_{k\in(S_x\cup S_y)^{(0)}} \left(\underbrace{|\tau_k(x) - \tau_k(y)|}_{\leq d(x,y) \text{ by 1-Lipschitzness}} + \frac{\tau_k(y)}{\bar{\tau}(y)} |\bar{\tau}(x) - \bar{\tau}(y)| \right) \\
&\leq \frac{\gamma r(m)}{\bar{\tau}(x)} \sum_{k\in(S_x\cup S_y)^{(0)}} \left(d(x,y) + \frac{\tau_k(y)}{\bar{\tau}(y)} |\bar{\tau}(x) - \bar{\tau}(y)| \right) \\
&= \frac{\gamma r(m)}{\bar{\tau}(x)} \sum_{k\in(S_x\cup S_y)^{(0)}} \left(d(x,y) + \frac{\tau_k(y)}{\bar{\tau}(y)} \left| \sum_{l\in(S_x\cup S_y)^{(0)}} (\tau_l(x) - \tau_l(y)) \right| \right)
\end{aligned}$$

[2] It is sufficient to only consider paths $\gamma_p : [0,1] \to \Sigma$ and such that $\dot{\gamma_p} = g(y) - g(x)$.

$$\leq \frac{\gamma r(m)}{\bar{\tau}(x)} \sum_{k \in (S_x \cup S_y)^{(0)}} \left(d(x,y) + \frac{\tau_k(y)}{\bar{\tau}(y)} \sum_{l \in (S_x \cup S_y)^{(0)}} |\tau_l(x) - \tau_l(y)| \right)$$
$$\leq \frac{\gamma r(m)}{\bar{\tau}(x)} \sum_{k \in (S_x \cup S_y)^{(0)}} \left(d(x,y) + \frac{\tau_k(y)}{\bar{\tau}(y)} \sum_{l \in (S_x \cup S_y)^{(0)}} d(x,y) \right)$$
$$\leq \frac{\gamma r(m)}{\bar{\tau}(x)} (2\dim(\Sigma)+1) \left(d(x,y) + (2\dim(\Sigma)+1) d(x,y) \right)$$
$$= \frac{\gamma r(m)}{\bar{\tau}(x)} (2\dim(\Sigma)+1)(2\dim(\Sigma)+2) d(x,y).$$

We summarize the result of the previous calculation:

$$d_C(g(x), g(y)) \leq \frac{\gamma r(m)}{\bar{\tau}(x)} (2\dim(\Sigma)+1)(2\dim(\Sigma)+2) d(x,y). \tag{3.3}$$

Let now D_n be such that $x \in D_n$ then $D_n \cap D_m \neq \emptyset$ by the previous claim and thus by point (5) of Lemma 3 we get:

$$\gamma^{-1} r(m) \leq r(n) = \tau_n(x) \leq \bar{\tau}(x). \tag{3.4}$$

We can now combine (3.3) with (3.4) and get

$$d_C(g(x), g(y)) \leq \gamma^2 (2\dim(\Sigma)+1)(2\dim(\Sigma)+2) d(x,y).$$

which shows the claim. ■

We have therefore constructed a Lipschitz map $g : X \to \Sigma$ with

$$\mathrm{Lip}(g) = \gamma^2 (2\dim(\Sigma)+1)(2\dim(\Sigma)+2) C_{\mathrm{q}},$$

where C_{q} is the smallest constant satisfying the conditions of quasi-convexity. This is what we wanted.

For the second statement, let $x \in \{\tau_k > 0\}$ for some k, then $g(x)(k) \neq 0$ and thus the simplex containing $g(x)$ must have v_k as a vertex. □

Lemma 5 *The dimension of* Σ *is independent of* ε *and* $\dim(\Sigma) \leq 2\dim_{AN}(Z) + 2$.

Proof This follows directly from point (4) of Lemma 3: The cover of X by sets $\{\tau_k > 0\}$ has multiplicity at most $2\dim_{\mathrm{AN}}(Z) + 3$. □

For the following discussion we need a short corollary which can be extracted from Theorem 2.4.2.

Corollary 8 *Let $n \geq 0$ and Y be a Lipschitz n-connected metric space and $Z \subseteq D^{n+1}$ a non-empty, closed subset of the closed $(n+1)$-ball. Then any Lipschitz map $f : Z \to Y$ can be extended to a Lipschitz map $\bar{f} : D^{n+1} \to Y$.*

Proof We know that $\dim_{\mathrm{AN}}(D^{n+1}) \leq n+1$ and it follows from Proposition 2 that $\dim_{\mathrm{AN}}(D^{n+1} \setminus Z) \leq \dim_{\mathrm{AN}}(D^{n+1})$. □

Notation 8 Denote the vertex set of a simplex $\Delta \subseteq \Sigma$ by $\mathcal{V}(\Delta) \subseteq \Sigma$. Furthermore denote the indices by $\mathcal{K}(\Delta) \subseteq \mathcal{K}(\Sigma)$.

Young's article [You14] contains a few problems which have subsequently been fixed in [You15]. In the following I proceed with the Lemmas and proofs given in the errata which replace the original version.

Lemma 6 (Lemma 1 in [You15]) *Let X be a Lipschitz n-connected metric space and let $Z \subseteq X$ be a non-empty closed subset of X. Suppose X_p with $p \in P$ are the connected components of $X \setminus Z$ and each ∂X_p is Lipschitz n-connected with uniformly bounded implicit constant. Then Z is Lipschitz n-connected as well.*

Proof Let $f : S^n \to Z$ be a Lipschitz continuous function.

Claim 3: There exists a Lipschitz extension $h : D^{n+1} \to Z$ of f with $\operatorname{Lip} h \lesssim \operatorname{Lip} f$.

Proof of Claim 3: By the Lipschitz n-connectivity of X we can extend f to a function $\bar{f} : D^{n+1} \to X$ such that $\operatorname{Lip} \bar{f} \lesssim \operatorname{Lip} f$. If $\bar{f}(D^{n+1}) \subseteq Z$ we have found a suitable h and we are done. Otherwise, for each $p \in P$ let $K_p := \bar{f}^{-1}(X_p)$.

We claim that $\bar{f}(\partial K_p) \subseteq \partial X_p$. Assume for a contradiction that there exists a $k_p \in \partial K_p$ such that $\bar{f}(k_p) \in \mathring{X}_p$. Then there exists a open neighbourhood of $\bar{f}(k_p)$ contained in X_p, the pre-image of this set under $\bar{f}$ is again an open set containing k_p which is a contradiction to the assumption that k_p was a boundary point. A similar argument works for points $\bar{f}(k_p) \notin \bar{X}_p$.

Our aim is to use Corollary 8. We restrict $\bar{f}$ to ∂K_p to define the functions

$$h_p := \bar{f}\Big|_{\partial K_p} : \partial K_p \to \partial X_p.$$

Each of these h_p, we can extend using the corollary to some $\bar{h_p} : K_p \to \partial X_p$ satisfying $\operatorname{Lip} \bar{h_p} \lesssim \operatorname{Lip} \bar{f}$. By construction we know that

$$\bar{h_p}\Big|_{\partial K_p} \equiv \bar{f}\Big|_{\partial K_p}.$$

Finally setting

$$h(x) := \begin{cases} \bar{f}(x), & \text{if } \bar{f}(x) \in Z \\ \bar{h_p}(x), & \text{if } \bar{f}(x) \in X_p \end{cases}$$

yields the desired Lipschitz extension h of f and because the implicit constant in the Lipschitz n-connectedness of the sets ∂X_p is uniformly bounded it follows that $\operatorname{Lip} h \lesssim \operatorname{Lip} f$. ■

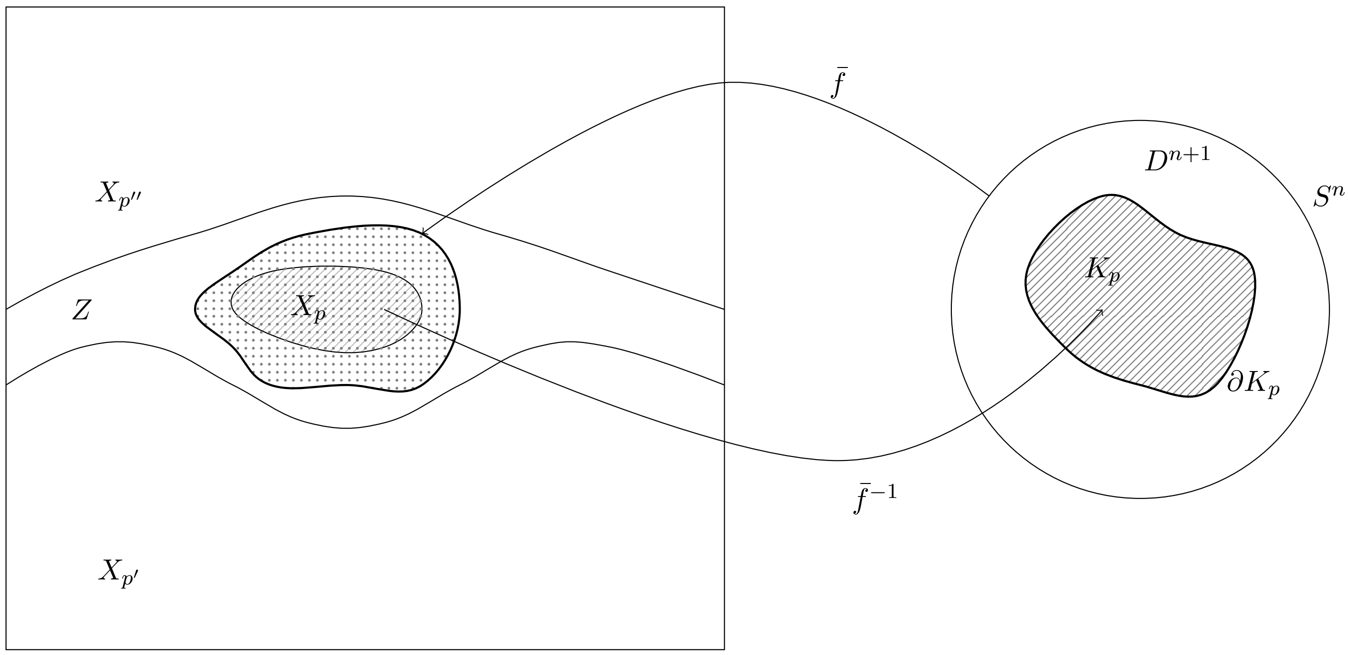

Figure 3.4: Illustration of the sets used for the proof of Lemma 6.

This shows the Lipschitz n-connectivity of Z. □

Lemma 7 (Lemma 2 in [You15]) *Under the assumptions of the main theorem (Theorem 2.4.1), there exists a Lipschitz map* $h : \Sigma^{(n+1)} \to Z$ *with Lipschitz constant independent of* ε *and such that* $d(h(g(z)), z) \lesssim \varepsilon$ *for all* $z \in Z$*, where* g *is the function that we defined in Lemma 4.*

Proof Define a function $h : \Sigma^{(0)} \to Z$ as follows: Let $v_k \in \mathcal{V}(\Sigma)$ be any vertex. If $D_k \cap Z \neq \emptyset$ then let $h(v_k) \in Z \cap D_k$ be some point (pick a choice). Otherwise, define $h(v_k) \in Z$ such that $d(h(v_k), \{\tau_k > 0\}) \leq 2d(Z, \{\tau_k > 0\})$.

Claim 4: The function $h : \Sigma^{(0)} \to Z$ is Lipschitz continuous with constant independent of ε.

Proof of Claim 4: Let v_k, $v'_k \in \Sigma$ be any two vertices that are connected by an edge e. Then by Lemma 3 we know that $r(k') \lesssim r(k) \lesssim r(k')$ and therefore $r(k') \lesssim l(e)$.[3] Without loss of generality assume that $r(k') \geq r(k)$. Then we can use point (1) and (2)

[3] $l(e) = \inf_\gamma \int_0^1 \sqrt{s(\gamma(t))^2 \cdot \dot\gamma(t)^2}\,\mathrm{d}t \geq \inf_\gamma \int_0^1 \sqrt{r(k') \cdot \dot\gamma(t)^2}\,\mathrm{d}t = \inf_\gamma r(k') \int_0^1 \left|\dot\gamma(t)^2\right|\mathrm{d}t = r(k') \int_0^1 |v_{k'} - v_k|\,\mathrm{d}t \gtrsim r(k')$

from Lemma 3 and calculate

$$\begin{aligned}
d(h(v_k),h(v_k')) &\le d(h(v_k),\{\tau_k>0\})+\operatorname{diam}(\{\tau_k>0\})+\operatorname{diam}(\{\tau_{k'}>0\})+d(\{\tau_{k'}>0\},h(v_{k'}))\\
&\lesssim d(Z,\{\tau_k>0\})+\operatorname{diam}(\{\tau_k>0\})+\operatorname{diam}(\{\tau_{k'}>0\})+d(\{\tau_{k'}>0\},Z)\\
&\lesssim d(Z,D_k)+\operatorname{diam}(\{\tau_k>0\})+\operatorname{diam}(\{\tau_{k'}>0\})+d(D_{k'},Z)\\
&\lesssim d(Z,D_k)+\operatorname{diam}(D_k)+r(k)+\operatorname{diam}(D_{k'})+r(k')+d(D_{k'},Z)\\
&\lesssim r(k)+r(k)+r(k)+r(k')+r(k')+r(k')\\
&\lesssim r(k')\lesssim l(e)
\end{aligned}$$

from which the claim follows. ■

By the previous lemma we know that Z is Lipschitz n-connected. Furthermore $\Sigma^{(0)}$ is a QC complex. Therefore we can inductively extend $h:\Sigma^{(0)}\to Z$ to a map $h:\Sigma^{(n+1)}\to Z$, which by abuse of notation we also call h.

Claim 5: This map satisfies the relation $d(h(g(z)),z)\lesssim\varepsilon$ for all $z\in Z$.

Proof of Claim 5: Let $z\in Z$ be some point. There exists a k such that $z\in D_k$. We use the triangle inequality and calculate

$$d(h(g(z)),z)\le d(h(g(z)),h(v_k))+d(h(v_k),z)\le \operatorname{Lip}(h)\cdot d(g(z),v_k)+d(h(v_k),z).$$

By Lemma 4 we know that $g(z)$ is in the star of v_k and also $r(k)=\varepsilon$, therefore $d(g(z),v_k)\lesssim\varepsilon$. Because $z\in Z\cap D_k$ we know that $h(v_k)\in Z\cap D_k$, so $d(h(v_k),z)\lesssim \operatorname{diam}(D_k)\lesssim r(k)=\varepsilon$. We conclude

$$d(h(g(z)),z)\lesssim \operatorname{Lip}(h)\cdot d(g(z),v_k)+d(h(v_k),z)\lesssim\varepsilon. \qquad ■$$

Thus we have constructed a Lipschitz map $h:\Sigma^{(n+1)}\to Z$ with the required properties.□

Notation 9 To indicate the particular ε used in the construction of Σ we may write $\Sigma(\varepsilon)$.

Lemma 8 *For any $k\in J$ we have $\operatorname{Lip}(g_k)\sim\varepsilon^{-1}$, where $g_k(x)=\frac{\tau_k(x)}{\bar\tau(x)}$ is the function we defined in Lemma 4.*

Proof We calculate

$$\operatorname{Lip}(g_k)=\sup_{x,y\in X}\frac{d(g_k(x),g_k(y))}{d(x,y)}=\sup_{x,y\in X}\frac{|g_k(x)-g_k(y)|}{d(x,y)}=\frac{\left|\frac{\tau_k(x)}{\bar\tau(x)}-\frac{\tau_k(y)}{\bar\tau(y)}\right|}{d(x,y)}.$$

This can be estimated as

$$\frac{\left|\frac{\tau_k(x)}{\bar\tau(x)}-\frac{\tau_k(y)}{\bar\tau(y)}\right|}{d(x,y)}\le\frac{\frac{r(k)}{\bar\tau(x)}}{r(k)}=\frac{1}{\bar\tau(x)}=\left(\sum_{k\in K}\tau_k(x)\right)^{-1}\sim((1+\dim(\Sigma))\varepsilon)^{-1}\sim\varepsilon^{-1}. \qquad □$$

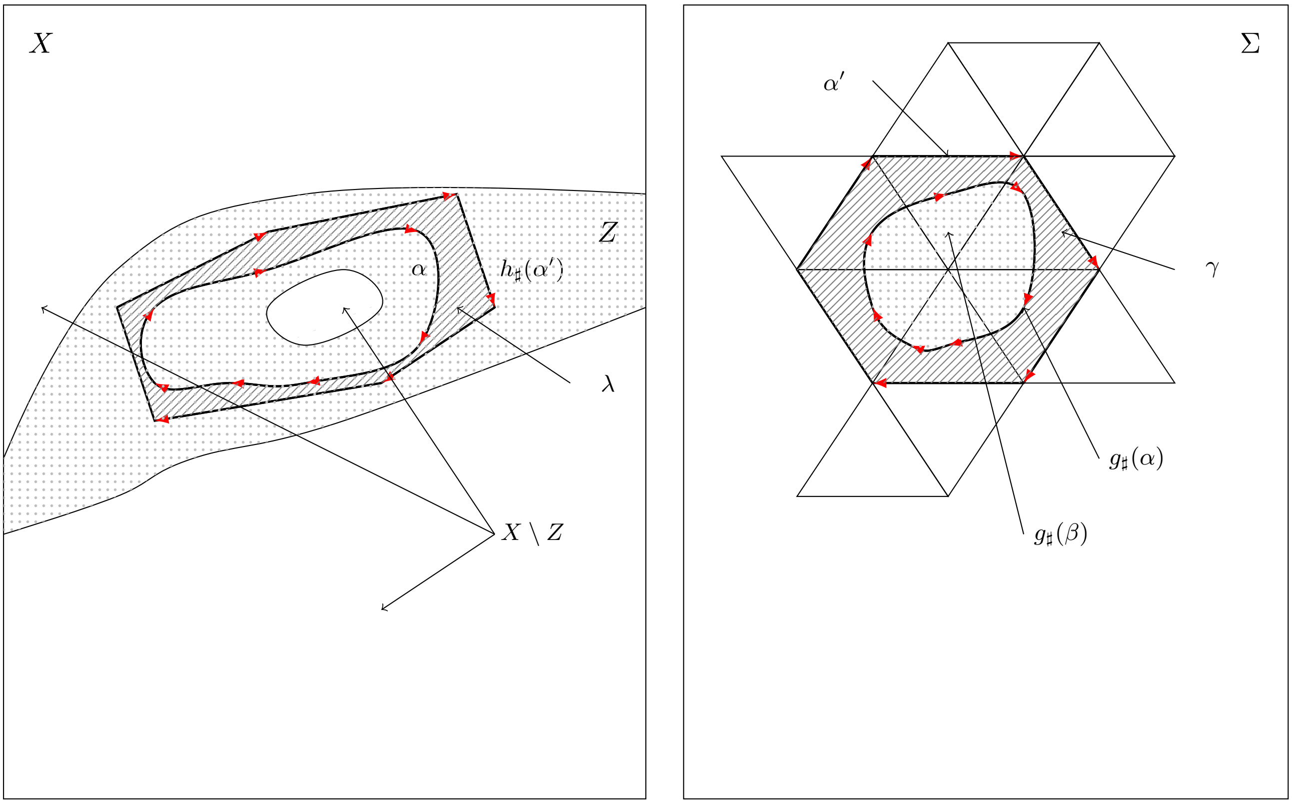

Figure 3.5: Illustration for the construction used in Lemma 9 and in the proof of the main theorem on page 34.

Remark 7 Note that, we have $\operatorname{diam}(\{\tau_k > 0\}) \sim \operatorname{diam}(D_k) + r(k) \sim \varepsilon$ for any $k \in J$.

We now have completed the necessary preparations to start with the proof of the main theorem. The proof follows almost directly from the following lemma. The construction and notations used in the lemma and proof are illustrated in Figure 3.5.

Lemma 9 (Lemma 3 in [You15]) *Assuming the conditions of the main theorem and using the functions g and h, which we defined in Lemma 4 and Lemma 7 respectively, let $\alpha \in C^L_m(Z)$ be a Lipschitz m-cycle in Z and $m \le n$. Then there exists a $c_\alpha > 0$, depending on the number of simplices in α and their Lipschitz constants such that for any $\varepsilon > 0$, there exist a simplicial m-cycle $\alpha' \in C_m(\Sigma(\varepsilon))$, and two annuli, $\gamma \in C^L_{m+1}(\Sigma(\varepsilon))$ and $\lambda \in C^L_{m+1}(Z)$ such that*

1. $\partial\gamma = g_\sharp(\alpha) - \alpha'$,
2. $\partial\lambda = \alpha - h_\sharp(\alpha')$,
3. $\operatorname{mass}\gamma \lesssim c_\alpha\varepsilon$,
4. $\operatorname{mass}\lambda \lesssim c_\alpha\varepsilon$.

Proof Let $L_g := \sup_{k \in J} \operatorname{Lip}(g_k)$. From Lemma 8 we know that $L_g \sim \varepsilon^{-1}$. Because α lies in Z we know that it is only covered by sets $\{\tau_k > 0\}$ with $k \in J$. Define $\delta_\Delta := \frac{1}{2(\dim(\Sigma)+1)L_g} \sim \varepsilon$ and subdivide the m-simplices of α into roughly δ_Δ^{-m} simplices each with diameter less than $\delta_\Delta < 1$. We write this subdivision as $\sum_{i=1}^{N} \alpha_i$ where $\alpha_i : \Delta^m \to Z$. There exists a constant $c_\alpha > 0$, depending only on the cycle α such that we can subdivide in a way that $N \leq c_\alpha \delta_\Delta^{-m}$ and because this results in a rescaling of the simplices we have furthermore that $\operatorname{diam}(\alpha_i) \leq \operatorname{Lip}(\alpha_i) < \delta_\Delta$ for each $i \in \{1, \dots, N\}$. Here $\operatorname{diam}(\alpha_i) := \operatorname{diam}(\alpha_i(\Delta^m))$.

For each point $z \in Z$ define $k(z) \in K$ such that $g_{k(z)}(z) \geq g_k(z)$ for all $k \in K$. Furthermore denote by $v(z) := v_{k(z)}$ the vertex $v \in \Sigma$ with the index $k(z)$. This is a (not necessarily unique) nearest vertex of $g(z)$ in Σ.

Claim 6: Given an $\alpha_i : \Delta^m \to Z$ in the subdivision and denoting the images under α_i of the vertices of Δ^m by $z_{i,0}, \dots, z_{i,m} \in Z$, then $v(z_{i,0}), \dots, v(z_{i,m}) \in \Sigma$ are the (not necessarily distinct) vertices of a simplex in Σ.

Proof of Claim 6: We know that

$$g_{k(z_{i,j})}(z_{i,j})(\dim(\Sigma) + 1) \geq \sum_{k \in K} g_k(z_{i,j}) = 1$$

and this implies

$$g_{k(z_{i,j})}(z_{i,j}) \geq \frac{1}{\dim(\Sigma) + 1}. \tag{3.5}$$

Furthermore note that we have

$$\operatorname{diam}(\alpha_i) \leq \operatorname{Lip}(\alpha_i) < \delta_\Delta = \frac{1}{2(\dim(\Sigma) + 1)L_g}$$

and therefore

$$d(z, z') < \frac{1}{2(\dim(\Sigma) + 1)L_g}$$

for any $z, z' \in \operatorname{im}(\alpha_i) \subset Z$. We can rewrite this as

$$L_g d(z, z') < \frac{1}{2(\dim(\Sigma) + 1)} \leq \frac{1}{\dim(\Sigma) + 1}.$$

Let $x \in \Delta^m$ be any point, then for $z := \alpha_i(x)$ we have

$$\frac{1}{\dim(\Sigma) + 1} - L_g d(z_{i,j}, z) > 0. \tag{3.6}$$

We further calculate

$$g_{k(z_{i,j})}(z_{i,j}) - L_g d(z_{i,j}, z) \leq g_{k(z_{i,j})}(z_{i,j}) - d(g_{k(z_{i,j})}(z_{i,j}), g_{k(z_{i,j})}(z)) \leq g_{k(z_{i,j}}(z)$$

and then combining this with (3.5) and (3.6) we get

$$g_{k(z_{i,j})}(z) \geq \frac{1}{\dim(\Sigma)+1} - L_g d(z_{i,j}, z) > 0. \tag{3.7}$$

This implies that $g_{k(z_{i,j})}(z) > 0$ for all $j \in \{0, \dots, m\}$ and therefore $\{v(z_{i,0}), \dots, v(z_{i,m})\}$ must be the vertex set of a simplex in Σ. ■

Claim 7: There exists a simplicial cycle $\alpha' \in C_m^{\mathrm{L}}(\Sigma)$ with $\operatorname{mass}\alpha' \lesssim c_\alpha$.

Proof of Claim 7: Define $\alpha' \in C_m^{\mathrm{L}}(\Sigma)$ to be the simplicial cycle $\alpha' := \sum_i \langle v(z_{i,0}), \dots, v(z_{i,m})\rangle$. Write each simplex in the sum as $\alpha_i' : \Delta^{n'(i)} \to \Sigma$. We can approximate the mass of the simplicial cycle by

$$\operatorname{mass}\alpha' \lesssim N\varepsilon^m \lesssim c_\alpha$$

because $\operatorname{diam}\alpha_i' \lesssim \varepsilon$. ■

It remains to construct γ and λ. Consider Σ as a subset of $\Delta_K = \{p : K \to [0,1] \mid \|p\|_1 = 1\}$ which we defined in the proof of Lemma 4. The set of vertices of Δ_K is $\{v_k\}_{k\in K}$. Denote by $\Delta^m = \langle e_0, \dots, e_m\rangle$ the standard m-simplex. Remember that for the subdivision of α we wrote $\alpha = \sum_i \alpha_i$ where $\alpha_i : \Delta^m \to Z$ and write $\alpha' = \sum_i \alpha_i'$ where $\alpha_i' : \Delta^{n'(i)} \to \Sigma$ is linear such that $\alpha_i'(e_j) = v(z_{i,j})$ for the simplicial cycle constructed above.

Let $x \in \Delta^m$ and let $z := \alpha_i(x) \in Z$.

Claim 8: $g(z)$ and $\alpha_i'(x)$ are both contained in the same simplex of Σ.

Proof of Claim 8: Let $s \in \Sigma$ and define by $\operatorname{supp}(s)$ the vertex set of the minimal simplex containing s. Note that

$$\operatorname{supp} g(z) = \{v_k \mid g_k(z) > 0\}$$

by the definition of g and Σ as the nerve of the covering. Furthermore we have that

$$\operatorname{supp}\alpha_i'(x) = \left\{v_{k(z_{i,0})}, \dots, v_{k(z_{i,m})}\right\}$$

by the Claim 6 above. From equation (3.7) it follows that $\operatorname{supp}(\alpha_i'(x)) \subseteq \operatorname{supp}(g(z))$. ■

Claim 9: There exists a $\gamma \in C_{m+1}^{\mathrm{L}}(\Sigma)$ with $\partial\gamma = g_\sharp(\alpha) - \alpha'$.

Proof of Claim 9: Let $\bar{\alpha}_i : \Delta^m \times [0,1] \to \Sigma$ be the homotopy defined by

$$\bar{\alpha}_i(x,t) := tg(\alpha_i(x)) + (1-t)\alpha_i'(x).$$

Note that we can identify $\Delta^m \times [0,1]$ with Δ^{m+1} and therefore $\gamma := \sum_i \bar{\alpha}_i$ can be viewed as a Lipschitz chain $\gamma \in C_{m+1}^{\mathrm{L}}(\Sigma)$ with $\partial\gamma = g_\sharp(\alpha) - \alpha'$. ■

Claim 10: $\operatorname{mass}\gamma \lesssim c_\alpha\varepsilon$.

Proof of Claim 10: By construction of the subdivison of α we know that $\mathrm{Lip}(\alpha_i) \lesssim \varepsilon$, furthermore we know that $\operatorname{mass} \alpha' \lesssim c_\alpha$ and because α' is a simplicial cycle with $\operatorname{diam}(\alpha_i') \lesssim \varepsilon$ we also have $\operatorname{Lip} \alpha_i' \lesssim \varepsilon$. We therefore have that $\operatorname{Lip} \bar{\alpha}_i \lesssim \varepsilon$ and further that $\operatorname{mass} \gamma \lesssim N\varepsilon^{m+1}$. From $N \lesssim c_\alpha \varepsilon^{-m}$ together with the above it follows that

$$\operatorname{mass} \gamma \lesssim c_\alpha \varepsilon.$$ ■

<u>*Claim 11:*</u> There exists a $\lambda \in C^{\mathrm{L}}_{m+1}(Z)$ such that $\partial\lambda = \alpha - h_\sharp(\alpha')$ and $\operatorname{mass} \lesssim c_\alpha \varepsilon$.

Proof of Claim 11: By Lemma 7 and $\operatorname{supp} \alpha_i'(x) \subseteq \operatorname{supp} g(\alpha_i(x))$ it follows that

$$d(h \circ \alpha_i', \alpha_i) \lesssim \varepsilon$$

and $\mathrm{Lip}(h \circ \alpha_i') \lesssim \varepsilon$. Let $p_i : \Delta^m \times \{0,1\} \to Z$ be defined as

$$p_i\big|_{\Delta^m \times 1} = h \circ \alpha_i'$$

and

$$p_i\big|_{\Delta^m \times 0} = \alpha_i,$$

then $\mathrm{Lip}(p_i) \lesssim \varepsilon$ and we can apply the Lipschitz n-connectivity of Z to extend

$$p_i : \Delta^m \times \{0,1\} = \Delta^m \times \partial[0,1] \to Z$$

to a map

$$p_i : \Delta^{m+1} = \Delta^m \times [0,1] \to Z.$$

Define $\lambda := \sum_i p_i$ by the previous identification. Then

$$\partial\lambda = \alpha - h_\sharp(\alpha')$$

and

$$\operatorname{mass} \lambda \lesssim N\varepsilon^{m+1} \lesssim c_\alpha \varepsilon.$$ ■

Proof (Proof of Theorem 2.4.1) Let $\beta \in C^{\mathrm{L}}_{m+1}(X)$ be a chain in X with $\partial\beta = \alpha$ and let γ and λ be as given by Lemma 9. Using the deformation theorem (Corollary 4) approximate $g_\sharp(\beta) - \gamma \in C^{\mathrm{L}}_{m+1}(\Sigma(\varepsilon))$ by a simplicial chain $P_{g_\sharp(\beta)-\gamma} := P(g_\sharp(\beta) - \gamma) \in C_{m+1}(\Sigma(\varepsilon))$. We know that $\partial P_{g_\sharp(\beta)-\gamma} = g_\sharp(\alpha) - g_\sharp(\alpha) + \alpha'$ by the properties of γ given in Lemma 9. Therefore $\partial P_{g_\sharp(\beta)-\gamma} = \alpha' \in C_m(\Sigma(\varepsilon))$. For $\lambda + h_\sharp(P_{g_\sharp(\beta)-\gamma}) \in C^{\mathrm{L}}_{m+1}$, we calculate

$$\partial(\lambda + h_\sharp(P_{g_\sharp(\beta)-\gamma})) = \alpha - h_\sharp(\alpha') + h_\sharp(P_{g_\sharp(\beta)-\gamma}) = \alpha - h_\sharp(\alpha') + h_\sharp(\alpha') = \alpha$$

and furthermore for the mass

$$\operatorname{mass}(\lambda + h_\sharp(P_{g_\sharp(\beta)-\gamma})) \lesssim c_\alpha\varepsilon + \operatorname{mass}(\beta - \gamma) \lesssim c_\alpha\varepsilon + \operatorname{mass}\beta.$$

Thus $\lambda + h_\sharp(P_{g_\sharp(\beta)-\gamma})$ is a chain in C^{L}_{m+1} with boundary α. Letting $\varepsilon \to 0$ we get

$$\operatorname{mass}(\lambda + h_\sharp(P_{g_\sharp(\beta)-\gamma})) \lesssim \operatorname{mass}\beta,$$

which is exactly

$$\mathrm{FV}_Z(\alpha) \lesssim \mathrm{FV}_X(\alpha)$$

as desired. □

Part III

Applications and Appendix

Chapter 4

Applications of the Theorem

> The laws of nature are constructed in such a way as to make the universe as interesting as possible.
>
> Freeman Dyson

We need the following theorem of Gromov.

Theorem 6 (Gromov [Wen07]) *Let X be a* $\mathbf{CAT(0)}$ *space. Then for any Lipschitz k-cycle $\alpha \in Z_k^L(X)$ in X the filling volume satisfies*

$$\mathrm{FV}_X^{k+1}(\alpha) \leq C \cdot \mathrm{mass}(\alpha)^{1+\frac{1}{k}}.$$

Where the constant C depends only on k.

Corollary 9 ([You14]) *Let X be a* $\mathbf{CAT(0)}$ *and let $Z \subset X$ be a non-empty closed subset of X with the metric given by the restriction of the metric of X and* $\dim_{AN}(X) < \infty$*, suppose that either one of the following conditions is true:*

- *Z is Lipschitz n-connected or,*
- *X is Lipschitz n-connected, and if X_p, $p \in P$ are the connected components of $X \setminus Z$, then the sets $H_p = \partial X_p$ are Lipschitz n-connected with uniformly bounded implicit constant.*

Then for any Lipschitz k-cycle $\alpha \in Z_k^L(Z)$ in Z with $k \leq n$ we have the relation

$$\mathrm{FV}_Z^{k+1}(\alpha) \leq C_1 \, \mathrm{FV}_X^{k+1}(\alpha) + C_1 \leq C_2 \, \mathrm{mass}(\alpha)^{1+\frac{1}{k}} + C_2.$$

Furthermore if $\delta_Z^{(k)}(x) := \sup\left\{\mathrm{FV}_Z^{k+1}(\gamma) \,\middle|\, \gamma \in Z_k^L(Z), \mathrm{mass}(\gamma) \leq x\right\}$ denotes the k^{th}-order Dehn function [BBD09], then

$$\delta_Z^{(k)}(x) \leq C_2 x^{1+\frac{1}{k}} + C_2.$$

Appendix A

Eilenberg–Steenrod Axioms for Homology

> For the things of this world cannot be made known without a knowledge of mathematics.
>
> Roger Bacon

Definition 16 (Eilenberg–Steenrod Axioms [Bre97]) Let $(H_n)_{n\in\mathbb{Z}} : (\mathbf{Top}, \mathbf{Top}) \to \mathbf{Ab}$ be a sequence of functors together with natural transformations $\partial : H_i(X, A) \to H_{i-1}(A)$ (where we use the notation $H_i(A) := H_i(A, \emptyset)$ and so on) such that the following axioms hold:

1. **Homotopy**: If $f : (X, A) \to (Y, B)$ is homotopic to $g : (X, A) \to (Y, B)$, then $f_* = g_* : H_*(X, A) \to H_*(Y, B)$,
2. **Exactness**: Each topological pair (X, A) together with inclusions $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$ induces a long exact sequence in homology
$$\cdots \xrightarrow{\partial} H_i(A) \xrightarrow{i_*} H_i(X) \xrightarrow{j_*} H_i(X, A) \xrightarrow{\partial} \cdots ,$$
3. **Excision**: Given a pair (X, A) and an open set $U \subseteq X$ such that $\bar{U} \subseteq \mathring{A}$, then the inclusion $\imath : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism
$$\imath_* : H_*(X \setminus U, A \setminus U) \xrightarrow{\sim} H_*(X, A),$$
4. **Dimension**: For a one point space **pt** we have $H_i(\mathbf{pt}) = 0$ for all $i \neq 0$,
5. **Additivity**: Given a disjoint union $X = \coprod_\alpha X_\alpha$ of a family of topological spaces $(X_\alpha)_{\alpha\in A}$, we have an isomorphism
$$H_i(X) \approx \bigoplus_\alpha H_i(X_\alpha).$$

Appendix B

Further Concepts that Are Used in the Text

> The whole thing that makes a mathematician's life worthwhile is that he gets the grudging admiration of three or four colleagues.
>
> Donald Knuth

We state some further concepts and basic results that are used throughout the text.

B.1 Additional Versions of the Deformation Theorem

Adapting the notation used by Young the deformation theorem can also be written as follows.

Theorem 7 (Deformation Theorem (Theorem 2 in [You06])) *Let X be a geodesic metric space, and let $(\tau, h : \tau \to X)$ be a triangulation of X. Then there is a constant $c = c(\tau)$ such that for all Lipschitz k-chains $\alpha \in C_k^L(X)$ with $\partial\alpha \in C_{k-1}(\tau)$ there exist $P_\tau(\alpha) \in C_k(\tau)$ and $Q_\tau(\alpha) \in C_{k+1}^L(X)$ such that:*

1. $\mathrm{mass}(P_\tau(\alpha)) \leq c \cdot \mathrm{mass}(\alpha)$,
2. $\mathrm{mass}(Q_\tau(\alpha)) \leq c \cdot \mathrm{mass}(\alpha)$,
3. $\partial Q_\tau(\alpha) = \alpha - P_\tau(\alpha)$.

For a simplicial complex, we have the following slightly more detailed version from Gruber.

Theorem 8 (Deformation Theorem (Theorem 2.1 in [Gru14])) *Given a simplicial complex Σ there is a constant $c > 0$ such that for any Lipschitz k-chain $\alpha \in C_k^L(\Sigma)$ there exist a simplicial k-chain $P(\alpha) \in C_k(\Sigma)$, a Lipschitz $(k+1)$-chain $Q(\alpha) \in C_{k+1}^L(\Sigma)$ and a Lipschitz k-chain $R(\alpha) \in C_k^L(\Sigma)$ such that*

1. $\text{mass}(P(\alpha)) \le c \cdot \text{mass}(\alpha)$,
2. $\text{mass}(Q(\alpha)) \le c \cdot \text{mass}(\alpha)$,
3. $\text{mass}(R(\alpha)) \le c \cdot \text{mass}(\partial\alpha)$,
4. $\partial Q(\alpha) = \alpha - P(\alpha) - R(\alpha)$,
5. $\partial R(\alpha) = \partial\alpha - \partial P(\alpha)$,
6. *$P(\alpha)$ and $Q(\alpha)$ are contained in the smallest subcomplex of Σ that contains α,*
7. *$R(\alpha)$ is contained in the smallest subcomplex of Σ that contains $\partial\alpha$.*

Corollary 10

- *If additionally α is a cycle then we have $R(\alpha) = 0$,*
- *If $\partial\alpha$ is simplicial, then we also have $R(\alpha) = 0$.*

B.2 Rademacher's Theorem

Theorem 9 (Rademacher's Theorem [Fed96]) *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz function. Then f is differentiable almost everywhere.*

B.3 Metric Differential

A similar result like Rademacher's Theorem holds in arbitrary metric space using the so called *metric differential.*

Theorem 10 ([Kir94]) *Let (X, d) be a metric space and let $f : \mathbb{R}^n \to X$ be a Lipschitz continuous function. Furthermore let $x \in S^{n-1}$ be any point. Then the limit*

$$\lim_{r \downarrow 0} \frac{d(f(p + rx), f(p))}{r}$$

exists for almost every point $p \in \mathbb{R}^n$.

Definition 17 ([Kir94]) We write

$$\text{MD}(f, p)(x) := \lim_{r \downarrow 0} \frac{d(f(p + rx), f(p))}{r}$$

whenever the above limit exists. $\text{MD}(f, p)(\cdot)$ is called the **metric differential** of f in the point p.

Theorem 11 ([Kir94]) *Let (X, d) be a metric space and let $f : \mathbb{R}^n \to X$ be a Lipschitz continuous function. Then for almost every point $p \in \mathbb{R}^n$ the metric differential $\text{MD}(f, p)(\cdot)$ is a seminorm on $\mathbb{R}^n$.*

Definition 18 ([Kir94]) Let (X, d) be a metric space and let $f : \mathbb{R}^n \to X$ be a Lipschitz continuous function. If for a point $p \in \mathbb{R}^n$ the metric differential $\mathrm{MD}(f, p)(\cdot)$ exists, then define the **Jacobian** of $\mathrm{MD}(f, p)(\cdot)$ as:

$$\mathcal{J}(\mathrm{MD}(f,p)) := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} n \left(\int_{S^{n-1}} \mathrm{MD}(f,p)(x)^{-n} \, \mathrm{d}\mathcal{H}^{n-1}(x) \right)^{-1}.$$

Remark 8 The following references provide more in depth information on the subject: [Kir94; KM03; Bon14; Gra14; Kar09; Mag10].

B.4 Čech Cohomology

Definition 19 A map between two simplicial complexes is called **simplicial map** if the images of the vertices of a simplex always span a simplex.[Mun]

Definition 20 Given a topological space X and an cover $\mathcal{U}$ of X by open sets, associate to it a simplical complex $\mathcal{N}(\mathcal{U})$ called the nerve of the cover $\mathcal{U}$. It has vertices v_α for each set $U_\alpha \in \mathcal{U}$ in the cover and any set of $k+1$ vertices spans a k-simplex if the $k+1$ sets in the cover corresponding to the vertices have non-empty intersection. Given another cover $\mathcal{V}$ which is a refinement of $\mathcal{U}$ (meaning that each $V_\alpha \in \mathcal{V}$ is contained in some $U_{\alpha'} \in \mathcal{U}$), then the associated inclusions induce a simplicial map $\imath : \mathcal{N}(\mathcal{U}) \to \mathcal{N}(\mathcal{V})$. Then the **Čech cohomology group** $\check{H}_i(X, G)$ is defined as the direct limit $\varinjlim H^i(\mathcal{N}(\mathcal{U}), G)$. [Hat02]

The idea in this construction is that a fine enough cover will yield an associated simplicial complex which is a good model of the space X.

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Nomenclature and Index

Nomenclature

$\bar{\mathcal{C}}$	The covering $\mathcal{C}$ with thickness increased by 2ε, page 21
$\bar{\tau}$	$\bar{\tau}(x) = \sum_{k\in K} \tau_k(x)$, page 25
Δ_K	The infinite simplex, page 25
$\dim_{\mathrm{AN}}(X)$	The Assouad-Nagata dimension of X, page 11
$\mathrm{FV}^n_X(\alpha)$	The filling volume of a Lipschitz n-cycle α in the space X, page 10
$\hat{\mathcal{B}}$	The subcover of $\mathcal{B}$ indexed by $\hat{I}$ and intersected with $X \setminus N_\varepsilon(Z)$, page 21
$\hat{I}$	$\{\hat{\imath} \in I \mid B_{\hat{\imath}} \not\subseteq N_\varepsilon(Z)\}$, page 21
$\mathrm{Lip}(f)$	The smallest possible constant L such that f is L-Lipschitz continuous, page 6
$\mathrm{mass}\,\alpha$	The mass of α, page 9
$\mathcal{B}$	The covering of $X \setminus Z$ as constructed by Lang and Schlichenmaier, page 18, 19
$\mathcal{C}$	cs-bounded covering given by finiteness of Assuad-Nagata dimension, page 21
$\mathcal{D}$	$\mathcal{D} := \hat{\mathcal{B}} \cup \mathcal{C}$, page 21
$\mathcal{K}(\Delta)$	The set of indices for the vertex set of the simplex Δ, page 28
$\mathcal{V}(\Delta)$	The vertex set of the simplex Δ, page 28
$\mathrm{MD}(f,p)(x)$	The metric differential of f in the point p in direction of x, page 44

$\mathrm{MD}(f,p)(x)$	The metric differential of f in the point p, page 44
∂	The boundary map, page 8
$\sqcup$	Disjoint sum, page 21
τ_k	The function $\tau_k(x) := \max\{0,\, r(k) - d(x, D_k)\}$, page 20
$B_n(X)$	$\operatorname{im}\partial_{n+1}$, page 9
$C_n(X)$	$C_n(X,\mathbb{Z})$, page 8
$C_n(X;G)$	The free abelian group with coefficients in G with basis the set of singular n-simplices in X, page 8
$C_n^{\Delta}(X)$	$C_n^{\Delta}(X,\mathbb{Z})$, page 8
$C_n^{\Delta}(X,G)$	The free abelian group with basis the open n-simplices of X and coefficients in G, page 8
$C_n^{\mathrm{L}}(X)$	$C_n^{\mathrm{L}}(X,\mathbb{Z})$, page 8
$C_n^{\mathrm{L}}(X;G)$	The free abelian group with coefficients in G with basis the set of singular Lipschitz n-simplices in X, page 8
D^d	The closed d-dimensional unit disk in $\mathbb{R}^d$ with the induced metric, page 7
$f \lesssim g$	There exists a constant $c > 0$ such that $f \leq cg$, page 7
$f \sim g$	$f \lesssim g$ and $g \lesssim f$, page 7
g	The Lipschitz map $g : X \to \Sigma$ defined by $g(x)(k) := \frac{\tau_k(x)}{\bar{\tau}(x)}$, page 24
h	The Lipschitz map $h : \Sigma^{(n+1)} \to Z$, page 29
$h^{(n)}$	The Lipschitz map $h : \Sigma^{(n)} \to Z$, page 29
I	The index set of the covering constructed by Lang and Schlichenmaier, page 18, 19
J	Index set for the covering $\mathcal{C}$, page 21
K	$K := \hat{I} \sqcup J$, page 21
$N_\varepsilon(Z)$	$N_\varepsilon(Z) := \{x \in X \mid d(x,Z) < \varepsilon\}$, page 20

r	$r(k) = \begin{cases} \delta \cdot d(D_k, Z), & k \in \hat{I} \\ \varepsilon, & k \in J \end{cases}$, page 20
S^d	The d-dimensional unit sphere in $\mathbb{R}^{d+1}$ with the induced metric, page 7
$Z_n(X)$	$\ker \partial_n$, page 9

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