The Boundary at Infinity of Gromov Hyperbolic Spaces

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ABSTRACT

We investigate the boundary at infinity of Gromov-hyperbolic metric spaces. The boundary of a given space is unique up to quasi-Möbius maps. We therefore first investigate which properties remain invariant under quasi-Möbius maps. In the second part we develop a new method to study the boundary at infinity by modifying the metric in such a way that we bring infinitely far points into a closed bounded space.

ZUSAMMENFASSUNG

Wir untersuchen den Rand im Unendlichen von Gromov-hyperbolischen metrischen Räumen. Da der Rand bis auf quasi-Möbius Abbildungen unabhängig von Basispunkt ist, untersuchen wir Eigenschaften welche invariant bleiben unter quasi-Möbius Abbildungen. Im zweiten Teil entwickeln wir eine neue Methode um den Rand im Unendlichen zu untersuchen. Dabei wird die Metrik so verändert, dass die Punkte im unendlichen einen endlichen Abstand zum Basispunkt bekommen.

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To Sara

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NOTATION

FREQUENTLY USED NOTATION

sn_κ	generalized sine function
cs_κ	generalized cosine function
$\mathrm{CAT}(\kappa)$	Cartan-Aleksandrov-Toponogov-spaces
$\mathrm{CBB}(\kappa)$	space with curvature bounded from below
M^n_κ	the n-dimensional model space of curvature κ
$\mathrm{cr}((x_1, x_2, x_3, x_4), d)$	cross-ratio of the four points (x_1, x_2, x_3, x_4) in the metric d
d_H	hyperbolic metric
d_E	Euclidean metric
d_{rad}	the radial part of the metric d with respect to a base point o
QM	quasi-Möbius
QS	quasi-symmetric
QI	quasi-isometric
i_p	metric inversion at point p
d_p	the metrized form of i_p
$\angle_p(x, y)$	the angle between the segments px and py
$\tilde{\angle}^\kappa_p(x, y)$	the angle between the segments px and py in the comparison space M^2_κ with constant curvature κ

1 INTRODUCTION

Damit das Mögliche entsteht, muß immer wieder das Unmögliche versucht werden.
— Hermann Hesse

The main objective of the thesis is to understand the boundary at infinity of a Gromov hyperbolic metric space. This is done in the following chapters, where Chapter 3 deals with the properties of spaces up to quasi-Möbius (QM) equivalence and states a uniformization theorem for quasi-Möbius spaces which characterizes spaces that are QM equivalent to a symbolic Cantor set. Chapter 4 develops a method to study the Gromov boundary by deforming the metric. The preliminary Chapter 2 is a prerequisite to both later chapters and gives a general overview of all the important theorems and definitions which are used throughout the thesis.

We give here a short introduction to the subject and state the main theorems of the thesis.

1.1 INVARIANT PROPERTIES OF QUASI-MÖBIUS MAPS

Given arbitrary metric spaces (X,d) and (Y,d'), a map $f : (X,d) \to (Y,d')$ is called *quasi-Möbius* (QM) if it is a homeomorphism and there exists a homeomorphism $\nu : [0,\infty[\to [0,\infty[$, such that for all quadruples $Q = (x_1,x_2,x_3,x_4)$ of distinct points of X and $Q' := (f(x_1),f(x_2),f(x_3),f(x_4))$ the following holds:

$$\mathrm{cr}(Q',d') \leq \nu(\mathrm{cr}(Q,d)).$$

Here

$$\mathrm{cr}(Q,d) := \frac{d(x_1,x_3)d(x_2,x_4)}{d(x_1,x_4)d(x_2,x_3)}$$

is the so called *cross-ratio*. In particular, the cross-ratio of a quadruple of points under the map f changes at most by something bounded by the *control function* ν.

Quasi-Möbius maps were first introduced in 1985 by Väisälä [Väi84] in order to study another class of maps called quasi-symmetric (QS) maps. Quasi-symmetric maps were also first studied by Tukia and Väisälä [TV80]. A quasi-symmetric map preserves (up to some control function) the ratio

$$\frac{d(x_1,x_2)}{d(x_1,x_3)},$$

and not just the cross-ratio. It is therefore a stronger property. If one wants to study extended metrics however - i.e., metrics where one considers also some point at infinity as part of the metric and the distance function can take values in $[0,\infty]$ - then quasi-symmetric maps are in some sense undesirable because they have to keep the infinitely remote point fixed. Quasi-Möbius maps solve this detail while sacrificing the ratio in favor of the cross-ratio.

Both QM and QS maps play an important role in metric analysis [Hei01], geometric group theory [DKN18] and the study of self-similarity [DS97].

In geometric group theory, when taking a finitely generated group G with generating set S, one can form the so called Cayley graph [Cay54] $\mathrm{Cay}(G,S)$ of G by taking G as the vertex set and edges between g and gs for every $g \in G$ and $s \in S$. One can then associate a metric to this graph by giving each edge the length 1. The Cayley graph of a group is independent of the generating set S up to quasi-isometry (roughly onto rough bi-Lipschitz maps) by the Švarc–Milnor lemma [Šva55; Mil68]. Quasi-isometry is a large scale notion of isometry. Meaning two spaces are quasi-isometric if they look the same from far away. Quasi-isometry is usually defined as follows: A map $f : (X,d) \to (Y,d')$ is called a *quasi-isometry* (QI) if there exist constants $\lambda \geq 1, C_0 \geq 0$ and $C_1 \geq 0$ such that

$$\frac{1}{\lambda}d(x,y) - C_0 \leq d'(f(x),f(y)) \leq \lambda d(x,y) + C_0,$$

for all $x, y \in X$ and for every $y \in Y$ there exists some $x \in X$ such that

$$d(f(x), y) \leq C_1.$$

The study of quasi-isometries goes back at least to Mostow [Mos68] who used them in order to proof the so called Mostow rigidity theorem, which states that two closed connected hyperbolic n-manifolds (for $n \geq 3$) are isometric if they are homotopy equivalent [Löh17].

In order to study spaces up to quasi-isometry, one needs to introduce large scale concepts of geometric notions. An important notion is Gromov hyperbolicity. A geodesic metric space is called Gromov δ-hyperbolic (for some $\delta \geq 0$), if in each geodesic triangle, the δ-neighborhoods of any two sides cover the third side [Gro87].[1] Examples of Gromov hyperbolic spaces are the hyperbolic space, metric trees and smooth simply connected manifolds with all sectional curvatures negative and bounded away from zero. Furthermore the fundamental groups of compact Riemannian manifolds with strictly negative sectional curvature are hyperbolic (meaning that their Cayley graphs are hyperbolic).

Quasi-isometries may not always be easy to understand. However if the space is Gromov hyperbolic then one can look at the Gromov boundary $\partial_\infty X$ of the space instead, that is the space of equivalence classes of infinite geodesic rays equipped with a specific metric [Gro87]. In order to define a metric, one introduces the Gromov product:

$$(x|y)_o = \frac{1}{2}(d(x,o) + d(y,o) - d(x,y)),$$

which is basically a measure of the distance for how far two geodesic rays stay close together. Two geodesic rays are then called equivalent if they stay close together infinitely, i.e., their Gromov product goes to $(x_i|y_i)_o \to \infty$ as $i \to \infty$. One can extend the Gromov product to the

1 This definition of δ-hyperbolicity with δ-slim triangles is generally credited to Eliyahu Rips. We will however mostly use Gromov's definition (introduced in detail in the preliminaries chapter) which works also in metric spaces which are not geodesic. For geodesic spaces both definitions are (up to a change of constant) equivalent.

Gromov boundary and use it to define a metric there. On the boundary one can work with quasi-symmetries instead of quasi-isometries. In fact quasi-symmetries on the boundary are in one to one correspondence with quasi-isometries in the interior [BS00]. Furthermore a Gromov hyperbolic group acts on the boundary of its Cayley graph by quasi-Möbius maps [Pau96; Bow98; MT10].

The study of QM maps is therefore justified by the fact that for a Gromov hyperbolic metric space (X, d) any two quasi-metrics on the boundary $\rho_b(\alpha, \beta) := a^{-(\alpha|\beta)_b}$, and $\rho_{b'} := a^{-(\alpha|\beta)_{b'}}$ with different base-points $b, b' \in X$ are related by QM maps: $\mathrm{id} : (\partial_\infty X, \rho_b) \to (\partial_\infty X, \rho_{b'})$ is a QM map. In particular the Gromov boundary up to change in base point can be characterized QM equivalent metric spaces [BS07].

We are therefore interested which properties apply to all QM equivalent spaces. Many interesting properties have been known to be invariant under quasi-symmetric maps but QM maps have not been investigated as strongly. We give a proof that the doubling property is invariant under QM maps. It has come to our attention later that this statement has been shown before in a different way. We also give a proof that the property of uniform disconnectedness is invariant under QM maps. This result has not been known before.

As an application we can now generalize a uniformization theorem of David-Semmes to QM spaces which gives a characterization of symbolic Cantor sets. In particular if a QM space is doubling, uniformly disconnected and uniformly perfect then it is a Cantor set.

In conclusion we get the following main theorems:

Theorem 1 (Invariance of doubling under quasi-Möbius maps)**.** *Let (X, d) be a doubling space. Let $f : (X, d) \to (Y, d')$ be a quasi-Möbius homeomorphism. Then (Y, d') is doubling.*

Theorem 2 (Invariance of uniform disconnectedness under quasi-Möbius maps)**.** *Let (X, d) be a metric uniformly disconnected space and let $f : (X, d) \to (Y, d')$ be a quasi-Möbius homeomorphism. Then (Y, d') is uniformly disconnected.*

Theorem 3. *Suppose that* (M,d) *is a complete, doubling, uniformly perfect and uniformly disconnected metric space. Then M is quasi-Möbius equivalent to a Cantor set.*

The proof of both invariance theorems follows a similar plan. First we realize that QM maps can be assembled from a quasi-symmetric map and up to two metric inversions. Because the results are already known for quasi-symmetric maps, it remains to show that they hold for the metric inversion as well. In case of the doubling property we give a direct proof constructing a covering. For uniform disconnectedness we give a proof by contradiction. Both proofs require to look at the infinitely remote points as a special point and handling this case separately.

1.2 METRIZING THE GROMOV CLOSURE

In Chapter 4 we develop a method to investigate the Gromov boundary by changing the metric in a specific way. This is a joint work based on an unpublished article [LS07] by Urs Lang and Viktor Schroeder.

Before we state the theorems we give the example which motivated this method. In classical geometry the hyperbolic space is usually constructed by taking an open unit ball in the euclidean space $(\mathbb{R}^n, \|\cdot\|)$ and then letting the metric on this ball be:

$$d_H(x,y) = \operatorname{arccosh}\left(1 + \frac{2\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\right).$$

This operation is reversible, and one can recover the euclidean metric from this space by calculating

$$\|x-y\| = \frac{\sinh(d_H(x,y)/2)}{\cosh(d_H(x,0)/2)\cosh(d_H(y,0)/2)}.$$

It is now reasonable to ask what happens if the same reverse construction is applied to a space with is reasonably close to hyperbolic. For example to a Gromov hyperbolic space as discussed previously, or a so called CAT(κ)-space. This is a geodesic space in which triangles are slimmer than corresponding triangles with the same side lengths in the simply connected 2-dimensional Riemannian manifold of constant sectional curvature κ. Examples of CAT(κ) spaces are simply-connected Riemannian manifolds of sectional curvature bounded above by κ. For $\kappa < 0$ we find that the same construction (with generalized trigonometric functions) still works and lets us study the boundary at infinity by applying the above formula, after which the boundary at infinity becomes the regular boundary with distance 1 from the base point. We can therefore complete the metric to study both the interior and the Gromov boundary. For $\kappa > 0$ we find a similar result for CBB(κ) spaces. Those are defined similarly to CAT(κ) spaces, but here we require the triangles to be fatter than the corresponding triangles in the comparison space. For Gromov δ-hyperbolic spaces we do not directly get a metric from the construction but only a semi-metric. However we can apply a standard construction which lets us recover the triangle inequality and therefore metricize this semi-metric. If $\delta < \ln(2)$ then the new metric is bi-Lipschitz equivalent to the semi-metric. For $\delta > \ln(2)$ this is no longer true as we show in a counter-example. For general metric spaces if we apply the method we still get a space which is topologically equivalent to the space we stared with. Furthermore points on the Gromov boundary are in one to one correspondence with points that have distance 1 from some base point in our new space.

We summarize the main results in simplified ($\kappa = -1$) form here:

Theorem 4. *Let $X = (X, d)$ be a complete CAT(κ)-space for $\kappa < 0$. Fix $o \in X$. Then*

1. *The function given by*

$$\rho_o(x, y) := \frac{\sinh(d(x, y)/2)}{\cosh(d(x, o)/2)\cosh(d(y, o)/2)}$$

is a metric on X.

2. *We can extend* ρ_o *to a metric on* $\bar{X} = X \cup \partial_\infty X$ *and the following relation holds for* $\xi, \eta \in \partial_\infty X$*:*

$$\rho_o(\xi, \eta) = 2\exp(-(\xi|\eta)_o).$$

3. *If* $\gamma : (X, d) \to (X, d)$ *is an isometry, then* $\gamma : (X, \rho_o) \to (X, \rho_o)$ *is a Möbius-map (Meaning it preserves the cross-ratio exactly).*

Theorem 5. *Let* (X, d) *be a complete intrinsic CBB*(1)*-space with* $\operatorname{diam}(X) < \pi$*, then*

$$\rho_o(x, y) := \frac{\sin(d(x, y)/2)}{\cos(d(x, o)/2)\cos(d(y, o)/2)}$$

is a metric.

We can generalize the result further to Gromov-hyperbolic spaces and get:

Theorem 6. *Let* (X, d) *be a* δ*-hyperbolic metric space with* $0 \leq \delta < \ln(2)$*. Then*

$$\rho_o(x, y) := \frac{\sinh(d(x, y)/2)}{\cosh(d(x, o)/2)\cosh(d(y, o)/2)}$$

is a semi-metric. Taking

$$\bar{\rho}_o(x, y) := \inf \sum_{i=0}^{n} \rho_o(x_i, x_{i+1}),$$

where the infimum runs over all finite chains of the form $x = x_0, \ldots, x_n = y$ *is a metric and* $\bar{\rho}_o \leq \rho_o \leq \lambda\bar{\rho}_o$ *for some* $\lambda \geq 1$*. The metric can be completed and the Cauchy completion* $(\bar{X} = X \cup \partial X, \bar{\rho})$ *coincides as a set with the Gromov boundary. Furthermore we have* $\omega \in \partial_\infty X$ *if and only if* $\bar{\rho}_o(o, \omega) = 1$*. For* $\delta > \ln(2)$*, there are spaces for which a* λ *as above does not exist, i.e., for which* ρ_o *and* $\bar{\rho})_o$ *are not bi-Lipschitz to each other.*

In a general metric space we can still extract some topological information from the construction:

Proposition 1. *For a general metric space the topologies of* (X, d) *and* $(X, \bar{\rho}_o)$ *are equivalent.*

Theorem 7. *Let* (X, d) *be a metric space with base point* $o \in X$ *and let* $x \in X \cup \partial X$ *be some point. Then for the completed metric* $\bar{\rho}_o$ *on* $\bar{X} = X \cup \partial X$ *the following holds:*

$$\bar{\rho}(o, x) = 1 \iff x \in \partial X.$$

2

PRELIMINARIES

The most merciful thing in the world, I think, is the inability of the human mind to correlate all its contents.

— H. P. Lovecraft

In this chapter we will introduce the main concepts and notions used in the rest of the thesis.

2.1 METRIC SPACES

In the following we use the convention that we call maps $d : X \times X \to \mathbb{R}$ *distance functions*.

Definition 1. *A set X together with a map $d : X \times X \to [0, \infty[$ is called* metric space *if the following conditions hold for all $x, y, z \in X$:*

1. $d(x, y) = 0 \iff x = y,$ (identity of indiscernibles)
2. $d(x, y) = d(y, x),$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z).$ (triangle inequality)

The function d is called the metric.

Quite often one works with a distance function that do not satisfy all the above conditions. In the following we introduce the most commonly used generalized metrics. Quite often the terminology used is not completely standardized across the literature [DD14; Väi05].

2.1.1 *Generalized Metrics*

If we relax the identity of indiscernibles we get a pseudo-metric space when only asking for one part of the implication and a meta-metric space in the other implication direction. Meta-metrics were first introduced by Väisälä [Väi05].

Definition 2. *A set X together with a map $d : X \times X \to [0, \infty[$ is called* pseudo-metric space *if the following conditions hold for all $x, y, z \in X$:*

1. $d(x, y) = 0 \Longleftarrow x = y,$
2. $d(x, y) = d(y, x),$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z).$ (triangle inequality)

The function d is called pseudo-metric.

Definition 3. *A set X together with a map $d : X \times X \to [0, \infty[$ is called* meta-metric space *if the following conditions hold for all $x, y, z \in X$:*

1. $d(x, y) = 0 \implies x = y,$
2. $d(x, y) = d(y, x),$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z).$ (triangle inequality)

The function d is called meta-metric.

If we generalize the triangle inequality, we get a quasi-metric.

Definition 4. *A set X together with a map $d : X \times X \to [0, \infty[$ is called* (K)-quasi-metric space *if there exists a $K \geq 0$ such that the following conditions hold for all $x, y, z \in X$:*

1. $d(x, y) = 0 \iff x = y,$ (identity of indiscernibles)
2. $d(x, y) = d(y, x),$ (symmetry)

3. $d(x,z) \leq K \max\{d(x,y), d(y,z)\}$.
(K-quasi-inequality / K-inframetric inequality)

The function d is called (K)-quasi-metric.

Remark 1. *Note that from* $d(x,y) + d(y,z) \leq 2\max\{d(x,y), d(y,z)\}$ *it follows that every metric space is also a* 2*-quasi-metric space. A* 1*-quasi-metric is called ultrametric. Every ultrametric is automatically a metric because* $\max\{d(x,y), d(y,z)\} \leq d(x,y) + d(y,z)$.

If we completely remove the triangle call the resulting distance function a semi-metric.

Definition 5. *A set X together with a map* $d : X \times X \to [0, \infty[$ *is called* semi-metric space *if the following conditions hold for all* $x, y, z \in X$*:*

1. $d(x,y) = 0 \iff x = y,$ (identity of indiscernibles)
2. $d(x,y) = d(y,x).$ (symmetry)

The function d is called semi-metric.

2.1.2 *Extended Metrics*

Sometimes it is useful to consider a map between two metric spaces with at most one extra point in each space representing the infinitely far away point.

Definition 6. *Let X be a set with cardinality at least 3. We call a map* $d : X \times X \to [0, \infty]$ *an* extended metric *on X if there exists a set* $\Omega(d) \subset X$ *with cardinality* 0 *or* 1 *and furthermore all of the following requirements are satisfied:*

1. $d_{|X\setminus\Omega(d)\times X\setminus\Omega(d)} : X \setminus \Omega(d) \times X \setminus \Omega(d) \to [0, \infty[$ *is a metric;*
2. $d(x, \omega) = d(\omega, x) = \infty$ *for all* $x \in X \setminus \Omega(d)$ *and* $\omega \in \Omega(d)$*;*
3. $d(\omega, \omega) = 0$ *for* $\omega \in \Omega(d)$.

If $\Omega(d)$ *is non empty we call* $\omega \in \Omega(d)$ *the* infinitely remote point *of X. By abuse of notation we may write* ∞ *for the point* ω.

2.1.3 *Properties of Metrics*

In order to study the notion of curvature it is helpful to introduce the notion of paths and path lengths on a metric space. This in turn allows us to define what a geodesic metric space is [Bus15; Bus18; Gro+99].

Definition 7. *Let (X, d) be a metric space and $x, y \in X$. A* path *from x to y is a continuous map*

$$\gamma : [0, 1] \to X$$

such that $\gamma(0) = x$ and $\gamma(1) = y$. The length *of a path is defined as follows:*

$$l(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) \,|\, n \in \mathbb{N},\, 0 = t_0 < t_1 < \ldots < t_n = 1 \right\}.$$

A path is called rectifiable *if it has finite length. The* induced intrinsic metric *of (X, d) is the function $d_I : X \times X \to [0, \infty]$ defined between two points $x, y \in X$ as the infimum of the lengths of all paths from x to y, and $d_I(x, y) = \infty$ if there is no finite path between x and y. If $d(x, y) = d_I(x, y)$ for all $x, y \in X$, then the space is called* length space *and the metric d is called* intrinsic.

Definition 8. *An intrinsic metric space (X, d) where for every two points $x, y \in X$, there exists a path γ with $l(\gamma) = d(x, y)$ is called* geodesic metric space *and γ is called a* geodesic between x and y.

Example 1. *If we remove any point from the Euclidean plane (with the Euclidean metric) the resulting space is no longer a geodesic space but it is still an length metric space.*

Definition 9. *A map $f : (X, d) \to (Y, d')$ is called* metric map *or* short map *if it does not expand distances between points. In mathematical terms for all points $x, y \in X$ it holds that:*

$$d'(f(x), f(y)) \leq d(x, y).$$

Definition 10. *Given a metric space (X, d) an infinite sequence of points $(x_i)_i$ in X is called* Cauchy sequence *if for every $\epsilon > 0$ there exists an integer $N > 0$ such that for all $i, j > N$, the distance satisfies*

$$d(x_i, x_j) < \epsilon.$$

A metric space (X, d) is called complete *if every Cauchy sequence in X converges to an element of X.*

We state the following well known construction in order to introduce some notation:

Proposition 2 (Completion of a Metric Space)**.** *Let (X, d) be a metric space. Then we can form the completion $\bar{X} = X \cup \partial X$ of X with respect to the metric d by applying the following procedure: Let $Cau(X, d)$ be the set of Cauchy sequences in X. We construct a pseudo-metric on $Cau(X, d)$ which by abuse of notation we also call d and which we define as follows:*

$$d(x, y) = \lim_{i \to \infty} d(x_i, y_i).$$

We now define an equivalence relation R on $Cau(X, d)$ by $x \sim y \iff d(x, y) = 0$. $\bar{X}$ is then the quotient $Cau(X, d)/R$ and d is a metric on $\bar{X}$. The space X is isometrically embedded by sending points to constant sequences $x \mapsto x_i = x$. We write ∂X to denote $\bar{X} \setminus X$.

2.2 NOTIONS OF CURVATURE

Here $(M^2_\kappa, \bar{d})$ denotes the complete, simply connected, Riemannian 2-manifold of constant sectional curvature[1] κ. The construction of model spaces is in more detail discussed in Section 4.2.

2.2.1 CAT(κ) Spaces

A space in which triangles are "slimmer" than the corresponding model triangles in a standard space of constant curvature κ is called CAT(κ). The terminology CAT(κ) goes back to Gromov [Gro87] and is an acronym for Élie Cartan, Aleksandr Danilovich Aleksandrov and Victor Andreevich Toponogov.

Definition 11. *A geodesic space (X, d) is said to satisfy the CAT(κ) inequality (i.e., X is a CAT(κ)-space) if, for all geodesic triangles Δ in X,*

$$d(p,q) \leq \bar{d}(\bar{p}, \bar{q})$$

for all comparison points $\bar{p}, \bar{q} \in \bar{\Delta} \subseteq M^2_\kappa$.

Example 2.

- *The Euclidean space is CAT(0).*
- *The hyperbolic space is CAT(-1).*
- *The unit sphere is CAT(1).*
- *Any CAT(κ) space is also CAT(κ') for all $\kappa' > \kappa$.*
- *Simply-connected Riemannian manifolds of sectional curvature bounded above by κ are CAT(κ) spaces.*

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1 In 2 dimensions this is the same as Gaussian curvature

2.2.2 *CBB(κ) Spaces*

The opposite condition that triangles are "fatter" than the model triangles is called CBB (curvature bounded from below). Often those spaces are also called *spaces with curvature $\geq \kappa$ in the sense of Alexandrov* [Top64; BGP92]. Those spaces were first introduced by Alexandrov in the 1950'.

Definition 12. *Let (X, d) be a metric space, $p, x, y \in X$ and $\kappa > 0$. Given the triangle p, x, y and the comparison triangle $\bar{p}, \bar{x}, \bar{y}$ in the 2 dimensional model space $(M^2_\kappa, \bar{d})$ with constant curvature κ, denote by $\bar{\measuredangle}^\kappa_p(x, y)$ the angle at $\bar{p}$ in the comparison triangle.*

The space (X, d) is called CBB(κ) if it is complete intrinsic and for all $p, x, y, z \in X$ it holds that

$$\bar{\measuredangle}^\kappa_p(x, y) + \bar{\measuredangle}^\kappa_p(y, z) + \bar{\measuredangle}^\kappa_p(z, x) \leq 2\pi.$$

We explicitly do not consider the following spaces to be CBB: circles of length greater than $\frac{2\pi}{\sqrt{\kappa}}$, line segments of length greater than $\frac{\pi}{\sqrt{\kappa}}$, the half-line $\mathbb{R}_+$, and the line $\mathbb{R}$.

Proposition 3 ([LP15; BBI01])**.** *A geodesic metric space (X, d) is CBB(κ) if and only if, for all geodesic triangles Δ in X,*

$$d(p, q) \geq \bar{d}(\bar{p}, \bar{q})$$

for all comparison points $\bar{p}, \bar{q} \in \bar{\Delta} \subseteq M^2_\kappa$.

Further equivalent definitions can be found in [AKP19].

Example 3.

- *Riemannian manifolds without boundary or with locally convex boundary whose sectional curvatures not less than κ [BGP92].*
- *Hilbert spaces are CBB(0)[BGP92].*
- *Given two CBB(κ) spaces with $\kappa \leq 0$, then their product is CBB(κ)[BBI01].*

- *Any CBB(κ) space is also CBB(κ') for all $\kappa' < \kappa$.*

An important result for metric spaces is the so called Alexandrov Lemma, which states:

Proposition 4 (Alexandrov's lemma [BH99; AKP19])**.** *Let $(M^2_\kappa, \bar{d})$ be one of the comparison spaces as above and let $x, y, z, o \in M^2_\kappa$ be distinct points with*

$$\bar{d}(o,x) + \bar{d}(x,y) + \bar{d}(y,z) + \bar{d}(z,o) < \begin{cases} 2\pi/\sqrt{\kappa}, \ \kappa > 0 \\ \infty, \ \kappa \leq 0 \end{cases}$$

and such that the geodesic ray through $[oy]$ crosses the segment $[xz]$ and $\measuredangle_y(o,x) + \measuredangle_y(z,o) \geq \pi$. Furthermore let $o', x', z' \in M^2_\kappa$ such that $\bar{d}(o,x) = \bar{d}(o',x')$, $\bar{d}(o,z) = \bar{d}(o',z')$ and

$$\bar{d}(x',z') = \bar{d}(x,y) + \bar{d}(y,z) < \begin{cases} \pi/\sqrt{\kappa}, \ \kappa > 0 \\ \infty, \ \kappa \leq 0 \end{cases} .$$

Geometrically the triangle (o', x', y') is a deformation of the first triangle (o, x, z) such that the two sides $[ox]$ and $[oz]$ are pushed apart and the geodesic segment $[x'z']$ now "contains" y. This is made clear as follows, let y' be the point on $[x'z']$ with $\bar{d}(x,y) = \bar{d}(x',y')$ (and therefore $\bar{d}(y,z) = \bar{d}(y',z')$). Then

$$\bar{d}(o,y) \leq \bar{d}(o',y').$$

2.2.3 *δ-Hyperbolic Metric Spaces*

In order to study groups, Gromov realized that the large scale geometry of negatively curved spaces can be characterized by a metric inequality [Gro87].

Definition 13 (Gromov product)**.** *Let (X, d) be a metric space. Then for points $x, y, o \in X$ the expression given by*

$$(x|y)_o := \frac{1}{2}\left(d(x,o) + d(y,o) - d(x,y)\right),$$

is called the Gromov product *of x and y with respect to the base point o.*

Definition 14 ((Gromov) δ-hyperbolicity)**.** *Given $\delta \geq 0$, a metric space (X, d) is called* (Gromov) δ-hyperbolic *if*

$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - \delta$$

for all $x, y, z, o \in X$. Or equivalently

$$d(x, o) + d(y, z) \leq \max\{d(x, y) + d(z, o), d(x, z) + d(y, o)\} + 2\delta$$

for all $x, y, z, o \in X$. A metric space (X, d) is called (Gromov) hyperbolic *if there exists a $\delta \geq 0$ such that X is δ-hyperbolic.*

Gromov hyperbolicity is a large scale property of a metric space. In particular it can not detect local properties of the space.

Example 4.

- *Every metric space of bounded diameter is hyperbolic.*
- *The hyperbolic plane $\mathbb{H}^2$ is $\ln(2)$-hyperbolic [NŠ16].*
- *Metric trees and real trees are 0-hyperbolic.*
- *For $\kappa < 0$, every CAT(κ)-space is hyperbolic.*
- *The Euclidean plane is **not** hyperbolic.*

2.3 LARGE SCALE GEOMETRY

2.3.1 *Quasi-Isometries*

Recall that a map between metric spaces is called bi-Lipschitz if it preserves distances up to some multiplicative constant. This means distances can be stretched or squeezed but not by too much. Formally we have the following:

Definition 15. *A map $f : (X,d) \to (Y,d')$ between two metric spaces is called* bi-Lipschitz *if there exists a constant $\lambda \geq 1$ such that for all $x, x' \in X$*

$$\frac{1}{\lambda} d(x,x') \leq d(f(x), f(x')) \leq \lambda d(x,x')$$

is satisfied.

Gromov hyperbolicity being a large scale property means it is not affected by a scaling of the metric by a multiplicative or additive constant. This allows us to define maps which keep the Gromov hyperbolicity intact but are strictly weaker than classical isometries. One needs a definition of a map which is roughly bi-Lipschitz in order to be compatible with the large scale geometry of metric spaces.

Definition 16 (Quasi-Isometry)**.** *A map $f : (X,d) \to (Y,d')$ between two metric spaces is called* quasi-isometric *if there exist constants $\lambda \geq 1$ and $C_0 \geq 0$ such that for all $x, x' \in X$ the following holds:*

$$\frac{1}{\lambda} d(x,x') - C_0 \leq d'(f(x), f(x')) \leq \lambda d(x,x') + C_0.$$

If the map is additionally roughly onto, meaning there exists a constant $C_1 \geq 0$ such that for every $y \in Y$ there exists a $x \in X$ with

$$d'(f(x), y) \leq C_1,$$

then f is called a quasi-isometry*. Two spaces are called* quasi-isometric *or* quasi-isometrically equivalent *if there exists a quasi-isometry between them.*

Example 5.

- *Every two metric spaces of bounded diameter are quasi-isometric.*
- *The Cayley graph of the group* $\mathbb{Z}^2$ *is quasi-isometric to the euclidean plane* $\mathbb{R}^2$.

We usually follow the convention to label maps which change the metric up to an additive constant by *"roughly-"*.

The following result is central to geometric group theory and also establishes the importance of quasi-isometries:

Proposition 5 (Švarc–Milnor Lemma [Šva55; Mil68])**.** *Let* X *be a length space. If* G *acts properly and cocompactly by isometries on* X*, then* G *is finitely generated, and for any choice of base point* $x_0 \in X$*, the map*

$$g \mapsto g \cdot x_0$$

is a quasi-isometry.

Definition 17. *The* Cayley graph $Cay(G, S)$ *of a finitely generated group* G *with generating set* S *is defined as the metric graph with vertices* G *and edges of length* 1 *between every* g *and* gs *for all* $g \in G$ *and* $s \in S$.

In particular it is easy to see that Cayley graphs for a groups G with generating sets S, S' are quasi-isometric.

2.4 THE BOUNDARY AT INFINITY

For the sake of completeness, in this section we sketch the main definitions and theorems necessary to explain the boundary at infinity. For a full treatment see [Gro87; Ham91; Bou96; BH99; BS07].

2.4.1 *The Boundary at Infinity of Hyperbolic Spaces*

Let (X, d) be a δ-hyperbolic metric space and $o \in X$ a base point. We will describe the boundary as an equivalence class of rays.

Definition 18 ((Gromov) boundary at infinity)**.** *A sequence of points* $\{x_i\} \subset X$ converges to infinity *if*

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty.$$

Two sequences $\{x_i\}$ *and* $\{y_i\}$ *that converge to infinity are* equivalent *if*

$$\lim_{i\to\infty} (x_i|y_i)_o = \infty.$$

The boundary at infinity $\partial_\infty X$ of X *is defined as the set of equivalence classes of sequences converging to infinity.*

2.4.2 *Quasi-Metrics on the Boundary at Infinity of Hyperbolic Spaces*

In order to equip the boundary at infinity with a metric, one first needs to extend the Gromov product to the boundary at infinity. One has to be careful in doing this, to make sure the resulting extension is well defined. In a second step one can then define a quasi-metric on the boundary. For suitable parameters one finally derives a metric from this quasi-metric.

Definition 19. *Let* (X, d) *be a* δ*-hyperbolic space and* $\partial_\infty X$ *its boundary at infinity. For two equivalence classes* $\xi, \gamma \in \partial_\infty X$ *of sequences converging to infinity we can define their* Gromov product *by*

$$(\xi|\gamma)_o = \inf \liminf_{i\to\infty} (x_i|y_i)_o,$$

where the infimum[2] *is taken over all sequences* $\{x_i\} \in \xi$ *and* $\{y_i\} \in \gamma$*.*

Remark 2 (Exercise 3.18 in [BH99] and [MT10])**.** *In case that* X *is a* $CAT(\kappa)$*-space (for any* κ*), the Gromov product can be extended to infinity by simply setting:*

$$(\lim x_i | \lim y_j)_o := \liminf_{i,j\to\infty} (x_i|y_j)_o.$$

Proposition 6 ([BS07])**.** *Given a* δ*-hyperbolic space* (X, d)*, a base point* $o \in X$ *and a fixed* $a > 1$*, then*

$$\rho(\xi, \gamma) := a^{-(\xi|\gamma)_o},$$

is a a^δ*-quasi-metric on* ∂_∞*.*

Remark 3. *If* (X, d) *is a proper* $CAT(\kappa)$*-space (for* $\kappa < 0$*), then* $\rho(\xi, \gamma) := a^{-(\xi|\gamma)_o}$ *is a metric on* $\partial_\infty X$ *for any* $a \in]1, e^{\sqrt{-\kappa}}]$ *and any base point* $o \in X$ *[Bou96].*

For $a^\delta \leq 2$ it is possible to apply a chain construction to get a metric out of the quasi-metric. In particular one can always construct the boundary in a way that it is metrizable.

Definition 20. *A metric* d *on* $\partial_\infty X$ *is called* visual metric *if there exists a base point* $o \in X$*,* $a > 1$ *and positive constants* c_1, c_2 *such that*

$$c_1 a^{-(\xi|\gamma)_o} \leq d(\xi, \gamma) \leq c_2 a^{-(\xi|\gamma)_o}$$

for all $\xi, \gamma \in \partial_\infty X$*.*

Remark 4. *This is the same as saying* $a^{-(\cdot|\cdot)_o}$ *is bi-Lipschitz to some metric on* $\partial_\infty X$*.*

Proposition 7. *Let* X *be a* δ*-hyperbolic space and let* $o \in X$ *be a base point. Then there exists a* $a_0 > 0$ *such that for every* $a \in]1, a_0]$*, there exists a metric* d *on* $\partial_\infty X$ *which is bi-Lipschitz to* $a^{-(\cdot|\cdot)_o}$ *[Ghy+90].*

---

2 In some literature sup lim inf some variation of this is used.

Proposition 8 (Application of Frink's Chain Construction[Fri37]). *Let (X, d) be a δ-hyperbolic space, fix $a > 0$ and a base point $o \in X$ and let $\rho : \partial_\infty X \times \partial_\infty X \to \mathbb{R}$ be given by $\rho(\xi, \gamma) = a^{(\xi|\gamma)_o}$. Then take*

$$d(\xi, \gamma) := \inf \sum_{i=0}^{n-1} \rho(\xi_i, \xi_{i+1}),$$

where the infimum is taken over all finite sequences $\xi = \xi_0, \dots, \xi_n = \gamma$ in $\partial_\infty X$. Then d is a metric and d is bi-Lipschitz to ρ.

Example 6.

- *The Gromov boundary of the real line consists of exactly two points.*
- *The Gromov boundary of the hyperbolic n-space is homeomorphic to the $(n-1)$-sphere.*
- *The Gromov boundary of (the Cayley graph of) a freely generated free group of rank at least 2 is a Cantor set.*

2.4.3 *Gromov Products Based at Infinity*

We used a fixed base point $o \in X$ in order to construct the boundary at infinity. But it is also possible to give a construction of the Gromov product on X with specifying a base point at infinity. In order to have a notion of the distance from a point at infinity ω to a point in X we can use Busemann functions. Busemann functions were first introduced in [Bus55]. Applying Busemann functions to base the Gromov product at infinity, we follow the treatment in [BS07].

Definition 21. *Given an extended metric space (X, d), a base point $o \in X$ and $\omega \in \Omega(X)$ we define:*

$$B_{\omega,o}(x) := \lim_i (d(x, \omega_i) - d(\omega_i, o)),$$

where $\omega_i \to \omega$ $(i \to \infty)$ is some sequence in X.

Given a δ-hyperbolic space (X, d) with $o \in X$ and $\omega \in \partial_\infty X$ we furthermore define

$$b_\omega(x, y) := (\omega|y)_x - (\omega|x)_y$$

as well as

$$b_{\omega,o}(\cdot) := b_\omega(\cdot, o).$$

Definition 22 (Gromov product based at Busemann function)**.** *Let X be a δ-hyperbolic space. Given $x, y, o \in X$ and $\omega \in \partial_\infty X$ let*

$$(x|y)_{\omega,o} := \frac{1}{2}\left(B_{\omega,o}(x) + B_{\omega,o}(y) - d(x, y)\right),$$

where $B_{\omega,o}(x) := \lim_{i\to\infty}(d(x, \omega_i) - d(\omega_i, o))$ is the limit of some sequence $\omega_i \to \omega$ $(i \to \infty)$. We can extend this to points on the boundary by setting:

$$(\xi|\gamma)_{\omega,o} := (\xi|\gamma)_o - (\omega|\xi)_o - (\omega|\gamma)_o.$$

Remark 5. *For a $CAT(\kappa)$ space with $\kappa < 0$, this is always well defined because all involved limits exist and are unique. For a δ-hyperbolic space one should replace the function B in the above definition with the function $b_{\omega,o}$ which is defined through the Gromov product and therefore also well defined. Note that*

$$(x|y)_{\omega,o} := \frac{1}{2}\left(B_{\omega,o}(x) + B_{\omega,o}(y) - d(x, y)\right) = (x|y)_o - (\omega|x)_o - (\omega|y)_o.$$

2.4.4 *Quasi Maps on the Boundary at Infinity*

We now introduce two important types of maps which appear naturally on the boundary at infinity.

2.4.4.1 *Quasi-Symmetric Maps*

A map which preserves the ratio is called quasi-symmetric. We precisely define those maps as follows. This definition goes back to Tukia and Väisälä [TV80] who first introduced them.

Definition 23. *We call a homeomorphism $f : (X,d) \to (Y,d')$ ν-*quasi-symmetric *if for all pairwise distinct $x_1, x_2, x_3 \in X$ we have*

$$\frac{d'(f(x_1),f(x_2))}{d'(f(x_1),f(x_3))} \leq \nu(\frac{d(x_1,x_2)}{d(x_1,x_3)}).$$

A homeomorphism $f : (X,d) \to (Y,d')$ is called quasi-symmetric *if it is ν-quasi-symmetric for some homeomorphism $\nu : [0,\infty[\to [0,\infty[$. It is called* symmetric *if for all pairwise distinct $x_1, x_2, x_3 \in X$ we have*

$$\frac{d'(f(x_1),f(x_2))}{d'(f(x_1),f(x_3))} = \frac{d(x_1,x_2)}{d(x_1,x_3)}.$$

Here (X,d) and (Y,d') are either metric or extended metric spaces. In case of an extended metric with $\omega \in \Omega(d)$ we take the following conventions:

- $\frac{d(x_1,\omega)}{d(x_1,x_3)} = \infty$,
- $\frac{d(x_1,x_2)}{d(x_1,\omega)} = 0$,
- $\frac{d(\omega,x_2)}{d(\omega,x_3)} = 1$.

2.4.4.2 *Quasi-Möbius Maps*

Quasi-Möbius maps were introduced by Väisälä in 1984[Väi84]. We follow the treatment given there.

Definition 24. *A map $f : (X,d) \to (Y,d')$ is* quasi-Möbius *if it is a homeomorphism and there exists a homeomorphism $\nu : [0,\infty[\to [0,\infty[$, such that for all quadruples $Q = (x_1,x_2,x_3,x_4)$ of distinct points of X and $Q' := (f(x_1),f(x_2),f(x_3),f(x_4))$,*

$$\mathrm{cr}(Q',d') \leq \nu(\mathrm{cr}(Q,d))$$

holds. Here the cross-ratio cr *is given by*

$$\mathrm{cr}(Q,d) := \frac{d(x_1,x_3)d(x_2,x_4)}{d(x_1,x_4)d(x_2,x_3)}.$$

(X,d) and (Y,d') are either metric or extended metric spaces. In case of an extended metric with ω the point at infinity the cross ratio is defined as:

$$\mathrm{cr}((x_1,x_2,x_3,\omega),d) := \frac{d(x_1,x_3)}{d(x_2,x_3)}.$$

The other cases are defined analogously. A map with $\mathrm{cr}(Q',d') = \mathrm{cr}(Q,d)$ *for all quadruples of distinct points is called* Möbius. *Whenever the metric is clear from the context we might write* $\mathrm{cr}(Q)$ *for* $\mathrm{cr}(Q,d)$.

Note that in particular all quasi-symmetric maps are also quasi-Möbius.

A quasi-Möbius map $f : (X,d) \to (Y,d')$ which keeps the infinitely remote point $\omega \in \Omega(d)$ fixed ($f(\omega) \in \Omega(d')$), has to preserve the ratio as well, and therefore is quasi-symmetric.

One motivation in the study of quasi-Möbius maps is that they give a useful tool while studying quasi-symmetric maps. In particular it is a more natural map when working with extended metrics because it does not require the infinitely remote point to be fixed.

Quite often one also encounters so called snowflake maps, which have been given this name in [BS00].

Definition 25 (Snowflake Map)**.** *A bijection $f : (X,d) \to (Y,d')$ between two metric spaces is called* snowflake map *if there exist constants $\lambda \geq 1$ and $\alpha > 0$ such that for all $x, x' \in X$ the following holds:*

$$\frac{1}{\lambda} d(x,x')^{\alpha} \leq d'(f(x),f(x')) \leq \lambda d(x,x')^{\alpha}.$$

2.4.4.3 *The Quasi-Möbius Structure on the Boundary at Infinity*

The justification of studying those maps comes from the following results.

Proposition 9 (5.2.8 in [BS07])**.** *Let (X,d) be a δ-hyperbolic space. Then there exists a constant $q \geq 1$ which depends only on δ, such that for any two* *quasi-metrics* $\rho_b := a^{-(\cdot|\cdot)_b}, \rho_{b'} := a^{-(\cdot|\cdot)_{b'}}$ *on* $\partial_\infty X$ *with the same parameter* $a > 1$*, and base points* $b, b' \in X \cup \{\infty\}$*, the identity map*

$$\mathrm{id} : (\partial_\infty X, \rho_b) \to (\partial_\infty X, \rho_{b'})$$

is quasi-Möbius.[3]

Proposition 10 (5.3.2 in [BS07])**.** *Let* (X, d) *be a boundary continuous hyperbolic space. Then for any two quasi-metrics* $\rho_b := a^{-(\cdot|\cdot)_b}, \rho_{b'} := a^{-(\cdot|\cdot)_{b'}}$ *on* $\partial_\infty X$ *with the same parameter* $a > 0$ *and base points* $b, b' \in X \cup \{\infty\}$*, the identity map*

$$\mathrm{id} : (\partial_\infty X, \rho_b) \to (\partial_\infty X, \rho_{b'})$$

is Möbius.

For hyperbolic groups, the induced action of the group on the boundary at infinity of its Cayley graph is by quasi-Möbius maps.

Proposition 11 ([Bow98; Pau96; MT10])**.** *Let G be a hyperbolic group (that is G is finitely generated, and the Cayley graph of G is hyperbolic). Then the induced action of G on* $\partial_\infty G$ *is by uniformly quasi-Möbius maps (i.e., all maps have the same control function), and it is cocompact on the space of distinct triples of points.*

Here cocompactness on triples means that there exists some constant $C > 0$ such that every triple (h_1, h_2, h_3) of distinct points in $\partial_\infty G$ can be mapped by some group element to a triple with pairwise distances greater than C.

Example 7. *If G is freely generated free group, then the Cayley graph of G is a tree, in particular it is hyperbolic. The boundary at infinity* $\partial_\infty G$ *of the Cayley graph is a Cantor set.*

Another important result establishes a one-to-one correspondence between quasi-isometric maps of Gromov hyperbolic spaces and quasi-symmetric maps on their boundaries.

3 In fact it is $(1, q)$-power-quasi-Möbius, which means that the control function of the map has the form: $\nu(t) = qt$.

Proposition 12 (Theorem 3.2.13 in [MT10]). *Let (X, d) and (Y, d') be Gromov hyperbolic spaces with uniform perfect Gromov boundaries $\partial_\infty X$ and $\partial_\infty Y$. Then each quasi-isometry $F : X \to Y$ induces a quasi-symmetric map $\partial_\infty F : \partial_\infty X \to \partial_\infty Y$ of the boundaries. Conversely, each quasi-symmetric map $f : \partial_\infty X \to \partial_\infty Y$ can be extended to a quasi-isometry $F : X \to Y$ such that $F_{|\partial X} = f$.*

Remark 6. *There are further correspondences [BS00; Jor10] for maps $f : (X, d) \to (Y, d')$ on visual Gromov hyperbolic spaces. We list them without going into the precise details of the definitions, as they are not used further in the thesis. We then have the following correspondences of maps $f : X \to Y$ and $\partial_\infty f : \partial_\infty X \to \partial_\infty Y$:*

1. *f is quasi-isometry $\leftrightarrow$ $\partial_\infty f$ is QS map with control function* $\nu(t) = \begin{cases} \lambda' t^{1/\alpha}, \ 0 < t < 1, \\ \lambda' t^{\alpha}, \ 1 \leq t. \end{cases}$
2. *f is rough similarity $\leftrightarrow$ $\partial_\infty f$ is snowflake map.*
3. *f is rough isometry $\leftrightarrow$ $\partial_\infty f$ is bi-Lipschitz map.*
4. *f is power quasi-isometry $\leftrightarrow$ $\partial_\infty f$ is power quasi-Möbius.*

2.4.5 *Metric Inversions*

Let (X, d) be an extended metric space and fix a point $p \in X$. Furthermore let

$$i_p(x, y) := \frac{d(x, y)}{d(x, p)d(y, p)}.$$

When $\omega \in \Omega(d)$, then $i_p(x, \omega) := \frac{1}{d(x,p)}$. This defines an extended quasi-metric on X. Applying the same chain construction as in Proposition 8:

$$d_p(x, y) := \inf \left\{ \sum_{i=1}^{k} i_p(x_i, x_{i-1}) \,\middle|\, x = x_0, \dots, x_k = y \in X \setminus \{p\} \right\},$$

results in a metric.

Definition 26. *This construction is called the* metric inversion*. When it is clear from the context we indicate the quasi-metric resulting from the inversion at the point p of the metric d by i_p and its metrization by d_p.*

The metric inversion satisfies the following equation:

$$\frac{1}{4} i_p(x, y) \leq d_p(x, y) \leq i_p(x, y) \leq \frac{1}{d(x, p)} + \frac{1}{d(y, p)}.$$

In particular $(X \setminus \{p\}, d_p)$ is a metric space. Furthermore the identity $\mathrm{id} : (X \setminus \{p\}, d) \to (X \setminus \{p\}, d_p)$ is a quasi-Möbius map [BK02; BHX08].

Remark 7. *It is easy to see that the map which sends the (extended) metric to its inversion $((X, d) \to (X, d_p), x \mapsto x)$ is a quasi-Möbius map. Calculating the cross-ratio we get:*

$$\frac{d_p(x_1, x_3)d_p(x_2, x_4)}{d_p(x_1, x_4)d_p(x_2, x_3)} \leq 8\frac{i_p(x_1, x_3)i_p(x_2, x_4)}{i_p(x_1, x_4)i_p(x_2, x_3)} = 8\frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}.$$

2.4.5.1 *Ptolemaic Spaces*

Definition 27 ([FS12])**.** *An extended metric space (X, d) is called* Ptolemy space*, if for all quadruples of points $x, y, z, w \in X$ the* Ptolemy inequality

$$d(x, z)d(y, w) \leq d(x, y)d(z, w) + d(x, w)d(y, z),$$

holds.

Remark 8. *A metric space* (X, d) *is Ptolemy if and only if for any* $p \in X$*, the metric inversion at the point* p *satisfies* $i_p = d_p$*.*

Proposition 13 ([FS11])**.** *Every* $CAT(0)$*-space satisfies the Ptolemy inequality.*

2.4.5.2 *Spherification*

There is a similar construction possible which is often called spherification and defined as follows for an extended metric space:

$$s_p(x,y) := \frac{d(x,y)}{(1+d(x,p))(1+d(y,p))},$$

with

$$s_p(x,\omega) := \frac{1}{1+d(x,p)},$$

for $\omega \in \Omega(d)$. Whereas the metric inversion sends the point p to infinity, the spherification transforms the space into a bounded space where all points have at most distance 1 from p. The function s_p can be metrized in the same way as in the case of the metric inversion [BHX08]. We write $\hat{d}_p$ for the metrized form. The following holds:

$$\frac{1}{4}s_p(x,y) \le \hat{d}_p(x,y) \le s_p(x,y) \le \frac{1}{1+d(x,p)} + \frac{1}{1+d(y,p)}.$$

In particular a map sending the (extended) metric d to $\hat{d}_p$ while keeping points fixed is a quasi-Möbius map.

2.4.6 *Quasi-Symmetric Invariants*

In order to study metric spaces one needs to define properties that help differentiate spaces from each other. The following properties are well know quasi-symmetric invariants of metric spaces.

2.4.6.1 *Doubling Property*

Definition 28. *We call a metric space* doubling with constant *D if every ball of finite radius can be covered by at most D balls of half the radius.*

Example 8.

- *The Euclidean space of any dimension as well as arbitrary subsets of it are doubling.*
- *The Heisenberg group with the Carnot metric is doubling.*

2.4.6.2 *Uniform Disconnectedness*

Definition 29. *For $\theta < 1$ we call a sequence of (at least 3 distinct) points $(x_0, x_1, \dots, x_n)$ in a metric space (X, d) a θ-*chain *if*

$$d(x_i, x_{i+1}) \leq \theta d(x_0, x_n)$$

holds for all $i \in \{0, 1, \dots, n-1\}$. A metric space is called uniformly disconnected with constant *θ if it contains no θ-chains.*[4] *A metric space (X, d) is called* uniformly disconnected *if there exists a $\theta < 0$ such that (X, d) is uniformly disconnected with constant θ.*

Uniform disconnectedness is a scale-invariant version of total disconnectedness.

Example 9.

- *The Cantor set is uniformly disconnected.*

4 And therefore also no θ'-chains for any $\theta' \leq \theta$.

- *The set* $\{\frac{1}{n} \mid n = 1, 2, \ldots\} \subset \mathbb{R}$ *is **not** uniformly disconnected.*

Remark 9. *A metric space* (X, d) *is uniformly disconnected if and only if there exists an ultra-metric* d' *on* X *which is bi-Lipschitz to* d [*DS97*].

2.4.6.3 *Uniform Perfectness*

Definition 30. *For* $C > 0$*, a metric space* (X, d) *is called* C-uniformly perfect *if for all points* $x \in X$ *and radii* $0 < r < \mathrm{diam}(X)$ *the annulus* $\overline{B}(x, r) \setminus B(x, Cr)$ *is non-empty. A metric space is called* uniformly perfect *if there exists a* $C > 0$ *such that the space is* C*-uniformly perfect.*

Example 10.

- *Every connected metric space is uniformly perfect.*
- *The Cantor set is uniformly perfect.*
- *Every Ahlfors regular space is uniformly perfect.*

2.4.6.4 *Invariance under Quasi-Symmetric Maps*

The properties of doubling, uniformly perfectness and uniformly disconnectedness are quasi-symmetric invariants.

Proposition 14 (Theorem 1.3.4 in [MT10])**.** *Let* $f : (X, d) \to (Y, d')$ *be a quasi-symmetric map onto another metric space. Then the following hold:*

1. *If* X *is doubling, then* Y *is doubling,*
2. *If* X *is uniformly perfect, then* Y *is uniformly perfect,*
3. *If* X *is uniformly disconnected, then* Y *is uniformly disconnected.*

3

SOME INVARIANT PROPERTIES OF QUASI-MÖBIUS MAPS

The truth may be puzzling. It may take some work to grapple with. It may be counterintuitive. It may contradict deeply held prejudices. It may not be consonant with what we desperately want to be true. But our preferences do not determine what's true.

— Carl Sagan

The following part has been published in [Hee17].

3.1 INTRODUCTION

Let (X, d) be a metric space. Recall that: X is *doubling* if there exists a constant $D > 0$, such that every ball of finite radius can be covered by at most D balls of half the radius. X is called *uniformly disconnected* if there exists a constant $\theta < 1$, such that X contains no θ-chain, i.e., a sequence of (at least 3 distinct) points $(x_0, x_1, \ldots, x_n)$ such that

$$d(x_i, x_{i+1}) \leq \theta d(x_0, x_n).$$

The aim of this chapter is to prove the following two theorems:

Theorem 8 (Invariance of doubling under quasi-Möbius maps)**.** *Let (X, d) be a doubling space. Let $f : (X, d) \to (Y, d')$ be a quasi-Möbius homeomorphism. Then (Y, d') is doubling.*

Theorem 9 (Invariance of uniform disconnectedness under quasi-Möbius maps)**.** *Let (X, d) be a metric uniformly disconnected space and let $f : (X, d) \to (Y, d')$ be a quasi-Möbius homeomorphism. Then (Y, d') is uniformly disconnected.*

The results are related to results of Lang-Schlichenmaier [LS05] and Xie [Xie08] who proved that quasi-symmetric maps respectively quasi-Möbius maps preserve the Nagata dimension of metric spaces. The present work has been inspired by the article of Xie [Xie08] and the work of Väisälä [Väi84]. We note that a space is doubling if and only if it has finite Assouad dimension [MT10]. However the Assouad dimension is not a quasi-symmetric (and therefore also not a quasi-Möbius) invariant [Tys+01].

We would like to note that we have been informed that Theorem 8 is a direct consequence of a published result of Li-Shanmugalingam [LS15].

It is well known that uniform disconnectedness is invariant under quasi-symmetric maps [MT10; DS97]. However its behavior under quasi-Möbius maps has not been studied before.

The proofs of both results use the fact that quasi-Möbius maps can be factorized into a quasi-symmetric and some number of metric inversions. Because we already know that the results hold for quasi-symmetric maps, one only needs to show that the properties stay invariant under the metric inversion.

The related property of uniform perfectness has already been shown to be invariant under the metric inversion in [Mey09]. It is therefore also invariant under quasi-Möbius maps.

In Section A.1 we prove a slight generalization of Theorem 8 and Theorem 9 for K-quasi-metric spaces.

3.2 INVARIANCE OF DOUBLING PROPERTY

3.2.1 *Preparations for the Proof*

For the proof we need the following proposition of Xie and a result of Väisälä which we cite verbatim.

Proposition 15 (Proposition 3.6 in [Xie08])**.** *Let $f : (X_1, d_1) \to (X_2, d_2)$ be a quasi-Möbius homeomorphism. Then f can be written as $f = f_2^{-1} \circ f' \circ f_1$, where f' is a quasi-symmetric map, and f_i for $i \in \{1, 2\}$ is either a metric inversion or the identity map on the metric space (X_i, d_i).*

Remark 10. *As a map between sets, f_1 is the identity on X_1, f_2 the identity on X_2 and $f' = f$. But from a metric point of view this partitions the map into two QM parts and a QS part.*

Note that while we used the result as stated in [Xie08], a factorization theorem already appears in [Väi84]. The result from [Väi84] requires an intermediate Banach space to apply the inversion however.

Proposition 16 (Theorem 3.10 in [Väi84])**.** *Let (X, d) be an unbounded metric space and let $f : X \to Y$ be a quasi-Möbius map. Then f is quasi-symmetric if and only if $f(x) \to \infty$ as $x \to \infty$. If X is any metric space and if $f : X \cup \{\infty\} \to Y \cup \{\infty\}$ is quasi-Möbius with $f(\infty) = \infty$, then the restriction $f_{|X}$ of f to X is quasi-symmetric.*

Remark 11. *Let (X, d) be an unbounded space. Then we can build the one-point completion with respect to the infinitely remote point $\bar{X} := X \cup \{\infty\}$ together with an extended metric $\bar{d}$. Let $\bar{d}(x, y) := d(x, y)$ and $\bar{d}(\infty, x) := \bar{d}(x, \infty) = \infty$ for all $x, y \in X$. Furthermore let $\bar{d}(\infty, \infty) = 0$. Then clearly (X, d) is doubling if and only if $(\bar{X}, \bar{d})$ is doubling. It then follows from [LS98] that the completion also has a doubling measure. Then Proposition 3.2 and Proposition 4.2 from [LS15] directly imply the following theorem:*

Theorem 10. *Let (X, d) be an metric doubling space with doubling constant D, where d is an extended metric [BS07] and denote by $\infty \in X$ the infinitely* *remote point in* (X, d)*. Furthermore let* $p \in X$ *with* $p \neq \infty$ *and let* i_p *be given by*

$$i_p(x,y) := \frac{d(x,y)}{d(p,x)\,d(p,y)},$$

for all $x, y \in X \setminus \{\infty\}$ *and*

$$i_p(\infty, x) := i_p(x, \infty) := \frac{1}{d(p,x)}.$$

Apply the chain construction to form a new extended metric[1] *from* i_p*:*

$$d_p(x,y) := \inf\left\{ \sum_{i=1}^{k} i_p(x_i, x_{i-1}) \,\middle|\, x = x_0, \dots, x_k = y \in X \setminus \{p\} \right\}.$$

Then (X, d_p) *is doubling with constant at most* $D^{10} + 1$*.*

Remark 12. *Note that if in addition d is Ptolemaic, then* $i_p = d_p$ *and in particular* (X, d_p) *is doubling with constant at most* $D^8 + 1$*.*

Before we were informed of the Li-Shanmugalingam result cited above we worked out a direct proof of the invariance of the doubling property under inversion. This proof is given here.

OUTLINE OF PROOF. The main idea of the proof is the following. We assume (X, d) is doubling and use this to construct a cover of a ball $B'_r(x_0)$ in the metric inversion of the space. The trick here is to take $B'_{\frac{1}{2}r}(\infty)$ as the first ball in the cover. This ensures that while approximating distances in (X, d) we have some fixed upper bounds that we can work with. The remaining points of the ball we can cover by some covering by the doubling property of (X, d). If we apply this multiple times we ensure that the covering balls in (X, d_p) all have diameter smaller than $\frac{1}{2}r$. Because this works quantitatively independent of the choice of radii and starting point x_0 we find a doubling constant.

1 Note that in the new metric p is the infinitely remote point and ∞ has finite distance to all points except to p.

Proof of Theorem 10. If (X, d) is bounded, consider the space $(\bar{X}, \bar{d})$, with $\bar{X} := X \cup \{\infty\}$ and $\bar{d}(x, y) := d(x, y)$ for all $x, y \in X$ and $\bar{d}(x, \infty) := \infty$. $(\bar{X}, \bar{d})$ is doubling. Furthermore if $(\bar{X}, \bar{d}_p)$ is doubling, then so is (X, d_p). We therefore only need to show the theorem for unbounded X.

We have the following relation for all $x, y \in X \setminus \{p\}$ [BHX08]:

$$\frac{1}{4} i_p(x, y) \leq d_p(x, y) \leq i_p(x, y) \leq \frac{1}{d(x, p)} + \frac{1}{d(y, p)}.$$

Let $x_0 \in X \setminus \{p\}$ and $r > 0$. Let $B' := B'_r(x_0) := \{x \in X \,|\, d_p(x_0, x) \leq r\}$ be the ball of radius r in the space (X, d_p). We consider the following two cases

1. If $B' \cap B'_{\frac{1}{2}r}(\infty) \neq \emptyset$, then let $A' := B'_r(x_0) \setminus B'_{\frac{1}{2}r}(\infty)$. Take $y_0 \in A'$. For any two points $x, y \in A'$ we have by definition of the metric d_p and the above relation that

$$i_p(x, y) = \frac{d(x, y)}{d(p, x)\, d(p, y)} \leq 4 d_p(x, y) \leq 8r,$$

and $\frac{1}{d(y,p)} = i_p(\infty, y) \geq d_p(\infty, y) > \frac{1}{2} r$. From this it follows that

$$d(x, y) \leq 8r d(p, x) d(p, y) \leq \frac{32}{r}.$$

In particular we know that $A' \subseteq B_{\frac{32}{r}}(y_0) := \{x \in X \,|\, d(y_0, x) \leq \frac{32}{r}\}$. By the assumption we furthermore have for all $x \in B'$ that

$$d_p(x, \infty) \leq 2r + \frac{1}{2} r = \frac{5}{2} r,$$

and therefore also

$$\frac{1}{d(p, x)} \leq \frac{5}{2} r,$$

from which it follows that

$$d(p, x) \geq \frac{2}{5r}.$$

The space (X,d) is doubling and we can find D^N balls b_i of radius $\frac{32}{r}2^{-N}$ with center points x_i covering $B_{\frac{32}{r}}(y_0)$. Let $\tilde{b}_i := b_i \cap A'$ then we have for all $x, y \in \tilde{b}_i$:

$$d_p(x,y) \leq i_p(x,y) = \frac{d(x,y)}{d(p,x)\,d(p,y)} \leq \frac{\frac{64}{r}2^{-N}}{\frac{2}{5r}\frac{2}{5r}} = \frac{64 \cdot 5^2 \cdot r^2}{2^2\,2^N\,r} = \frac{400}{2^N}\,r.$$

In particular for $N := 10$ we know that we have constructed a cover of $B' \subseteq A' \cup B'_{\frac{1}{2}r}(\infty)$ by $D^{10}+1$ balls of radius $\frac{1}{2}r$.

2. In case that $B' \cap B'_{\frac{1}{2}r}(\infty) = \emptyset$, we know that $d_p(x_0,\infty) > r$ and also $d_p(B',\infty) := \inf_{x\in B'} d_p(x,\infty) \geq \frac{1}{2}r$. For all $y \in B'$ we have

$$i_p(x_0,y) = \frac{d(x_0,y)}{d(p,x_0)\,d(p,y)} \leq 4d_p(x_0,y) \leq 4r,$$

from which it follows that

$$d(x_0,y) \leq 4rd(p,x_0)d(p,y) \leq \frac{4r}{d_p(\infty,x_0)\,d_p(\infty,y)} \leq \frac{4r}{d_p(\infty,B')^2}.$$

We therefore have $B' \subseteq B_{\frac{4r}{d_p(\infty,B')^2}}(x_0)$ and by the doubling property of (X,d) we can cover by D^N balls b_i of radius $\frac{4r}{d_p(\infty,B')^2}\,2^{-N}$ with center points x_i. Let $\tilde{b}_i := b_i \cap B'$, then we have for any two $x, y \in \tilde{b}_i$:

$$\begin{aligned} d_p(x,y) \leq i_p(x,y) &= \frac{d(x,y)}{d(p,x)d(p,y)} \\ &\leq \frac{\frac{8r}{d_p(\infty,B')^2}\,2^{-N}}{d(p,x)\,d(p,y)} \\ &= 2^{-N+4}\frac{d_p(\infty,x)\,d_p(\infty,y)}{d_p(\infty,B')^2}r. \end{aligned}$$

Furthermore we have

$$d_p(x,\infty) \le d_p(x_0,x)+d_p(x_0,\infty) \le r+d_p(B',\infty)+r \le 5d_p(B',\infty).$$

In conclusion we get that

$$2^{-N+4}\frac{d_p(\infty,x)\, d_p(\infty,y)}{d_p(\infty,B')^2} \le 2^{-N+4}\frac{5^2 d_p(\infty,B')^2}{d_p(\infty,B')^2} = \frac{8\cdot 5^2}{2^N}.$$

It therefore follows that if we take $N := 9$, then we have a covering of B' by D^9 balls of radius $\frac{1}{2}r$.

□

3.2.2 *Proof of the Invariance of Doubling under quasi-Möbius Maps (Theorem 8)*

Proof of Theorem 8. It remains to show the theorem for (X,d) being a doubling metric space, $f : (X,d) \to (X,d')$ a metric inversion and we have the following cases to check:

1. (X,d) unbounded, (X,d') bounded;
2. (X,d) and (X,d') both unbounded but with different points at infinity.

Case 2 follows directly from Theorem 10. In the situation of 1, d' is a metric inversion d_p where p is an isolated point in X. That is there exists a $\epsilon > 0$ such that $d(p,x) > \epsilon$ for all $x \in X \setminus \{p\}$. The proof of Theorem 10 still holds. □

3.3 INVARIANCE OF UNIFORM DISCONNECTEDNESS

The proof of Theorem 9 will again make use of some of the propositions from the previous sections.

OUTLINE OF PROOF. Similar to the proof of the doubling property, the infinitely remote point plays a special role in the proof. We use the fact that it has finite distance in one space but not the other. We show that if one space has a chain, then the other space must have a chain as well. And we construct such a chain directly by taking a detour over the inversion point p.

In the following let (X, d) be a metric space, $p \in X$ and $\theta \leq \frac{1}{32}$. We assume that (X, d_p) is not θ-uniformly disconnected, in particular there is some θ-chain $(x_0, x_1, \dots, x_n)$ in $(X \setminus \{p\}, d_p)$. We keep this notation for the rest of this section. In addition we introduce the following notation for convenience: Let $r_i := d(p, x_i)$, $l := d(x_0, x_n)$ and $l_i := d(x_i, x_{i+1})$. This is illustrated in Figure 3.1. Without loss of generality we can assume $r_n \geq r_0$.

Remark 13. *The condition for $(x_0, x_1, \dots, x_n)$ being a θ-chain in (X, d_p) implies that*

$$\frac{l_i}{r_i r_{i+1}} \leq \frac{4\theta l}{r_n r_0} \; \forall i \in \{0, \dots, n-1\}.$$

On the other hand if

$$\frac{l_i}{r_i r_{i+1}} \leq \frac{\theta l}{4 r_n r_0} \; \forall i \in \{0, \dots, n-1\}$$

holds, then $(x_0, x_1, \dots, x_n)$ is a θ-chain in (X, d_p).

Lemma 1. *Assume that (X, d) contains no $\sqrt[3]{4\theta}$-chains. Then there is an index $s \in \{0, \dots, n-1\}$ such that*

$$l_s > l\sqrt[3]{4\theta}$$

and

$$\max\{r_s, r_{s+1}\}\sqrt[3]{4\theta} \geq r_0.$$

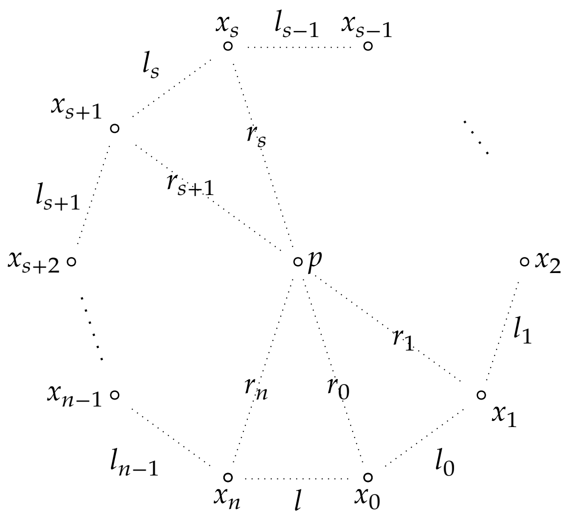


Figure 3.1: The view of the θ-chain in (X, d)

Proof. Assume for a contradiction that $r_s \sqrt[3]{4\theta} < r_0$ and $r_{s+1} \sqrt[3]{4\theta} < r_0$. Then from the condition in the remark above it follows

$$\frac{l_s}{r_s r_{s+1}} \leq \frac{4\theta l}{r_n r_0} < \frac{4\theta l_s}{\sqrt[3]{4\theta} r_n r_0} < \frac{4\theta l_s}{\sqrt[3]{4\theta}^3 r_s r_{s+1}} = \frac{l_s}{r_s r_{s+1}} \tag{3.1}$$

which is a contradiction. □

Proposition 17. *(X, d) contains a $\sqrt[3]{4\theta}$-chain.*

Proof. By the previous lemma we know that there must be some index q such that $r_q \sqrt[3]{4\theta} \geq r_0$ and for all $i \in \{0, \ldots, q-1\}$ we have that $r_i \sqrt[3]{4\theta} < r_0$.

We claim that $(x_q, x_{q-1}, \ldots, x_1, x_0, p)$ is a $\sqrt[3]{4\theta}$-chain in (X, d). If this were not so, there would be some $i \in \{0, \ldots, q-1\}$ for which $r_q \sqrt[3]{4\theta} < l_i$. But then

$$\frac{r_q \sqrt[3]{4\theta}^2}{r_0 r_q} < \frac{r_q \sqrt[3]{4\theta}}{r_i r_q} \leq \frac{r_q \sqrt[3]{4\theta}}{r_i r_{i+1}} < \frac{l_i}{r_i r_{i+1}} \leq \frac{4\theta l}{r_n r_0} \tag{3.2}$$

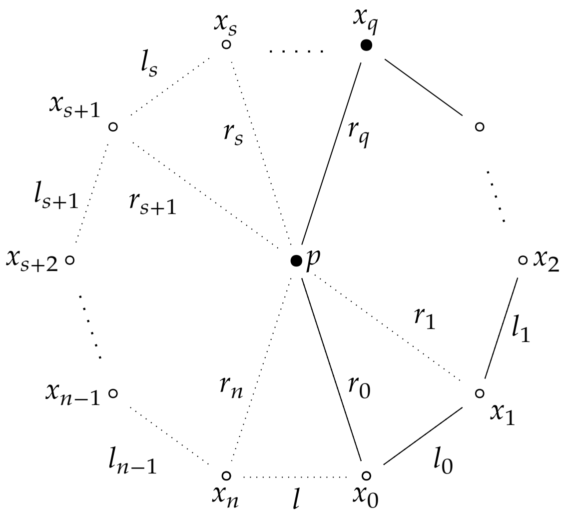


Figure 3.2: The constructed $\sqrt[3]{4\theta}$-chain in (X, d)

implies

$$r_n < \sqrt[3]{4\theta} l \leq \frac{1}{2} l, \tag{3.3}$$

which is a contradiction to the triangle inequality of the metric space (X, d). □

Proof of Theorem 9. The proof of the theorem now follows directly from Proposition 15. □

3.4 APPLICATIONS OF THE THEOREMS

For the following we need a short definition [DS97]: Let F be a finite set with $k \geq 2$ elements and let F^∞ denote the set of sequences $\{x_i\}_{i=1}^\infty$ with $x_i \in F$. For two elements $x = \{x_i\}, y = \{y_i\} \in F^\infty$ let

$$L(x,y) = \sup\{I \in \mathbb{N} \mid \forall 1 \leq i \leq I : x_i = y_i\}.$$

In particular we have $L(x,x) = \infty$ and $L(x,y) = 0$ if $x_1 \neq y_1$. Given $0 < a < 1$ set $\rho_a(x,y) = a^{L(x,y)}$. This defines an ultrametric on F^∞. We call (F^∞, ρ_a) the *symbolic k-Cantor set with parameter a*.

A more geometric way of looking at this definition is the following. Consider a infinite complete binary metric tree (T,d) where each edge has length 1. This is clearly a 0-hyperbolic space and in particular CAT(-1). Label the root vertex by o. Then F^∞ coincides with the Gromov boundary $\partial_\infty T$ of T and L is precisely the Gromov product $(\cdot|\cdot)_o$ on $\partial_\infty T$.

As an application of the theorems we provide a generalization of the following result by David and Semmes:

Proposition 18 (Proposition 15.11 (Uniformization) in [DS97])**.** *Suppose that (M,d) is a compact metric space which is bounded, complete, doubling, uniformly disconnected, and uniformly perfect. Then M is quasi-symmetrically equivalent to the symbolic Cantor set F^∞, where we take $F = \{0,1\}$ and we use the metric ρ_a on F^∞ with parameter $a = \frac{1}{2}$.*

We can generalize this result as follows:

Theorem 11. *Suppose that (M,d) is a complete, doubling, uniformly perfect and uniformly disconnected metric space. Then M is quasi-Möbius equivalent to the symbolic Cantor set as given above.*

Proof. Let $p \in M$ be some point and let $s_p(x,y) = \frac{d(x,y)}{(d(x,p)+1)(d(y,p)+1)}$. Let

$$\hat{d}_p(x,y) = \inf\left\{ \sum_{i=1}^{k} s_p(x_i, x_{i-1}) \,\middle|\, x = x_0, \dots, x_k = y \in X \right\},$$

then we have by [BK02; BHX08] that

$$\frac{1}{4}s_p(x,y) \leq \hat{d}_p(x,y) \leq s_p(x,y) \leq \frac{1}{1+d(x,p)} + \frac{1}{1+d(y,p)}.$$

Then the space $(M, \hat{d}_p)$ is bounded and satisfies all the properties of the above proposition: The map $f : (X,d) \to (X, \hat{d}_p)$ given by $d \mapsto \hat{d}_p$ is Möbius. By Theorem 9 and Theorem 8, doubling and uniformly disconnectedness are invariant under Möbius maps. The invariance of uniformly perfectness follows from [Mey09], and the invariance of completeness follows from [BS15]. Totally boundedness follows from the doubling property and therefore the space $(X, \hat{d}_p)$ is compact. □

Definition 31 (Ahlfors regularity [Hei01]). *A metric space (X,d) admitting a Borel regular measure μ such that*

$$C^{-1}r^n \leq \mu(B_r) \leq Cr^n$$

for some constants $C \geq 1, n > 0$ and for all closed balls B_r of radius $0 < r <$ diam(X)*, is called* Ahlfors regular.

We can apply the same idea to Proposition 16.9 in [DS97] and we get:

Corollary 1. *Let (M,d) be a complete Ahlfors regular metric space of dimension γ which is uniformly disconnected. Then there exists a doubling measure μ on F^∞, and (M,d) is quasi-Möbius equivalent to (F^∞, D), where D is given by*

$$D(x,y) = \left(\mu(\bar{B}(x, d_a(x,y))) + \mu(\bar{B}(y, d_a(x,y)))\right)^{\frac{1}{\gamma}},$$

and $0 < a < 1$.

This follows from the above remarks and the invariance of Ahlfors regularity under $d \mapsto \bar{d}_p$ as shown in [LS15].

4

METRIZING THE GROMOV CLOSURE

To me the converging objects of the universe perpetually flow,
All are written to me, and I must get what the writing means.

— Walt Whitman

The following chapter is based on the unpublished article "Metrizing the Gromov closure" by Urs Lang and Viktor Schroeder [LS07]. The original article only considered the cases CAT(-1) and δ-hyperbolic spaces and some proof were left incomplete. I extended the work to general CAT(κ) spaces and CBB(κ) spaces and gave some results that apply to general metric spaces. The CBB result also answers a question originally asked by Marc Lischka. Together with Viktor Schroeder I constructed an example space with proves a sharp bound on the constant used in one of the results. Results that have been taken from [LS07] are cited as such.

4.1 INTRODUCTION AND DEFINITIONS

The main idea of this article is to provide a method to study the boundary at infinity of a hyperbolic metric space by transforming the metric in such a way that the boundary appears as the points on a sphere with fixed radius from a chosen base point. This will in particular give a better understanding of the metric structures on the ideal boundary of a CAT(-1) space (which are due to Hammenstädt [Ham91] and Bourdon [Bou96]).

4.1.1 *Generalized Trigonometric Functions*

In the following we use the following trigonometric functions.

Definition 32. *Let*

$$\mathrm{sn}_\kappa(x) := \begin{cases} \sin(\sqrt{\kappa}x)/\sqrt{\kappa}, & \text{if } \kappa > 0 \\ x, & \text{if } \kappa = 0 \\ \sinh(\sqrt{-\kappa}x)/\sqrt{-\kappa}, & \text{if } \kappa < 0 \end{cases}$$

and

$$\mathrm{cs}_\kappa(x) := \begin{cases} \cos(\sqrt{\kappa}x), & \text{if } \kappa > 0 \\ 1, & \text{if } \kappa = 0 \\ \cosh(\sqrt{-\kappa}x), & \text{if } \kappa < 0. \end{cases}$$

Also let the inverse functions be given by

$$\mathrm{arcsn}_\kappa(x) := \begin{cases} \arcsin(\sqrt{\kappa}x)/\sqrt{\kappa}, & \text{if } \kappa > 0 \\ x, & \text{if } \kappa = 0 \\ \mathrm{arcsinh}(\sqrt{-\kappa}x)/\sqrt{-\kappa}, & \text{if } \kappa < 0 \end{cases}$$

and

$$\mathrm{arccs}_\kappa(x) := \begin{cases} \arccos(x)/\sqrt{\kappa}, & \text{if } \kappa > 0 \\ 1, & \text{if } \kappa = 0 \\ \mathrm{arccosh}(x)/\sqrt{-\kappa}, & \text{if } \kappa < 0. \end{cases}$$

Those functions are the solutions of the second order differential equation $f'' + \kappa f = 0$ satisfying the initial conditions

$$\mathrm{sn}_\kappa(0) = 0, \quad \mathrm{sn}'_\kappa(0) = 1, \qquad \mathrm{cs}_\kappa(0) = 1, \quad \mathrm{cs}'_\kappa(0) = 0.$$

See [BC91, pp. 170–222] and [LS14] for a more in depth treatment of those functions.

Proposition 19 (Trigonometric Identities [LS14])**.** *The following classical identities hold for the generalized functions:*

1. $\mathrm{cs}_\kappa(x)^2 + \kappa\,\mathrm{sn}_\kappa(x)^2 = 1$, $x \in \mathbb{R}$,
   (Pythagorean trigonometric identity)

2. $\operatorname{sn}_\kappa(x+y) = \operatorname{sn}_\kappa(x)\operatorname{cs}_\kappa(y) + \operatorname{cs}_\kappa(x)\operatorname{sn}_\kappa(y)$ *for* $x, y \in \mathbb{R}$,

3. $\operatorname{cs}_\kappa(x+y) = \operatorname{cs}_\kappa(x)\operatorname{cs}\kappa(y) - \kappa\operatorname{sn}_\kappa(x)\operatorname{sn}_\kappa(y)$ *for* $x, y \in \mathbb{R}$,

4. $\operatorname{arcsn}_\kappa(x) = \ln(x + \sqrt{x^2+1})/\sqrt{-\kappa}$ *for* $\kappa < 0, x \in \mathbb{R}$,

5. $\operatorname{arccs}_\kappa(x) = \ln(x + \sqrt{x^2-1})/\sqrt{-\kappa}$ *for* $\kappa < 0, x \in [1, \infty[$,

6. $\operatorname{cs}_\kappa(\operatorname{arccs}_\kappa(x)/2) = \frac{\sqrt{x+1}}{\sqrt{2}}$ *for* $\kappa < 0, x \in [-1, \infty[$.

4.2 THE SPACES OF CONSTANT CURVATURE

Although the following are standard mathematical results we could not find a similar complete construction in the literature. This section is based on hand written notes of Viktor Schroeder. We construct the model spaces and derive formulas for the distance calculation which we will use later on.

In the following let $0 < n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$. We define models M_κ^n of the space of constant curvature κ as subsets of $\mathbb{R}^{1+n}$. We use coordinates $x = (x_0, x_1, \dots, x_n)$ and define the bilinear form:

$$\langle x, y\rangle_\kappa = x_0 y_0 + \kappa \sum_{i=1}^{n} x_i y_i.$$

Let M_κ be the connected component of $e_0 = (1, 0, \dots, 0)$ of the set $\{x \in \mathbb{R}^{1+n} \mid \langle x, x\rangle_\kappa = 1\}$. M_κ is a sub-manifold of $\mathbb{R}^{1+n}$ and for every $p \in M_\kappa$ the tangent space at p is given by

$$T_p M_\kappa = \{v \in \mathbb{R}^{1+n} \mid \langle v, p\rangle_\kappa = 0\}.$$

In particular $T_{e_0} M_k = \mathbb{R}^n \subset \mathbb{R}^{1+n}$, where $\mathbb{R}^n \hookrightarrow \mathbb{R}^{1+n}$, $x \mapsto (0, x)$ is the canonical embedding. The bilinear form $\frac{1}{\kappa}\langle\cdot,\cdot\rangle_\kappa$ is positive definite[1] on $T_p M_\kappa$ and defines a Riemannian metric on M_κ.

---

1 For $\kappa \geq 0$ this is trivial. For $\kappa < 0$ consider the following: From $\langle p, p\rangle_\kappa = 1$ it follows that $p_0^2 = 1 - \kappa \sum_{i=1}^n p_i^2$. Furthermore $\langle p, v\rangle_\kappa = 0$ implies that $v_0 = -\frac{\kappa \sum_{i=1}^n p_i v_i}{p_0}$. Therefore we get

$$\begin{aligned}
\frac{1}{\kappa}\langle v, v\rangle_\kappa &= \frac{\kappa(\sum_{i=1}^n p_i v_i)^2}{p_0^2} + \sum_{i=1}^{n} v_i^2 \\
&= \frac{\kappa(\sum_{i=1}^n p_i v_i)^2}{1 - \kappa \sum_{i=1}^n p_i^2} + \sum_{i=1}^{n} v_i^2 \\
&= \frac{\kappa((\sum_{i=1}^n p_i v_i)^2 - (\sum_{i=1}^n p_i^2)(\sum_{i=1}^n v_i^2)) + \sum_{i=1}^n v_i^2}{1 - \kappa \sum_{i=1}^n p_i^2}
\end{aligned}$$

We can now apply the Cauchy-Schwarz inequality to get

$$\frac{1}{\kappa}\langle v, v\rangle_\kappa \geq 0.$$

In the case of M_0, $T_pM_0 = \mathbb{R}^n \subset \mathbb{R}^{1+n}$ for all p and there we view $\frac{1}{\kappa}\langle\cdot,\cdot\rangle_\kappa|_{\mathbb{R}^n} = \langle\cdot,\cdot\rangle$ as the standard inner product on $\mathbb{R}^n$.

Let $v \in T_{e_0}M_\kappa$, i.e. $v \in \mathbb{R}^n$ with $\|v\| = 1$ and consider

$$\gamma_v(t) = \mathrm{cs}_\kappa(t)e_0 + \mathrm{sn}_\kappa(t)v$$

then $\gamma_v(t) \in M_\kappa$ since

$$\langle\gamma_v(t), \gamma_v(t)\rangle_\kappa = \mathrm{cs}_\kappa(t)^2 + \kappa\,\mathrm{sn}_\kappa(t)^2 = 1.$$

We calculate

$$\begin{aligned} \langle\gamma_v, \gamma_v'\rangle_\kappa &= 0 \quad | \text{ since } \gamma_v' = -\kappa\,\mathrm{sn}_\kappa\, e_0 + \mathrm{cs}_\kappa\, v, \\ \frac{1}{\kappa}\langle\gamma_v', \gamma_v'\rangle_\kappa &= \frac{1}{\kappa}(\kappa^2\,\mathrm{sn}_\kappa^2 + \kappa\,\mathrm{cs}_\kappa) = 1, \\ \gamma_v'' &= (-\kappa)\gamma_v \implies \gamma_v''(t) \perp_\kappa T_{\gamma_v(t)}M_\kappa. \end{aligned}$$

Thus γ_v is the unit speed geodesic in M_κ with $\gamma_v(0) = e_0, \gamma_v'(0) = v$.

Consider the linear map ("rotation in the plane span(e_0, v)") given by

$$R_v(t) = \begin{pmatrix} \mathrm{cs}_\kappa(t) & -\kappa\,\mathrm{sn}_\kappa(t) \\ \mathrm{sn}_\kappa(t) & \mathrm{cs}_\kappa(t) \end{pmatrix},$$

i.e. given in an orthonormal basis $e_0, v, v_2, \ldots, v_n$ it looks like

$$\begin{pmatrix} \begin{array}{c|} R_v(t) \\ \hline \end{array} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

This is a 1-parameter group of isometries of M_κ, since it leaves the bilinear form $\langle\cdot,\cdot\rangle_\kappa$ invariant.

The isometry group of M_κ is generated by the $R_v(T)$, $v \in S^{n-1}$ and $O(n)$, the orthogonal group embedded as

$$\left(\begin{array}{c|c} 1 & \\ \hline & O(n) \end{array}\right) \subset \mathrm{Gl}(1+n).$$

These isometries are the linear maps of $\mathbb{R}^{1+n}$ leaving $\langle\cdot,\cdot\rangle_\kappa$ invariant and preserve the connected component M_κ. IsoM_κ is transitive.

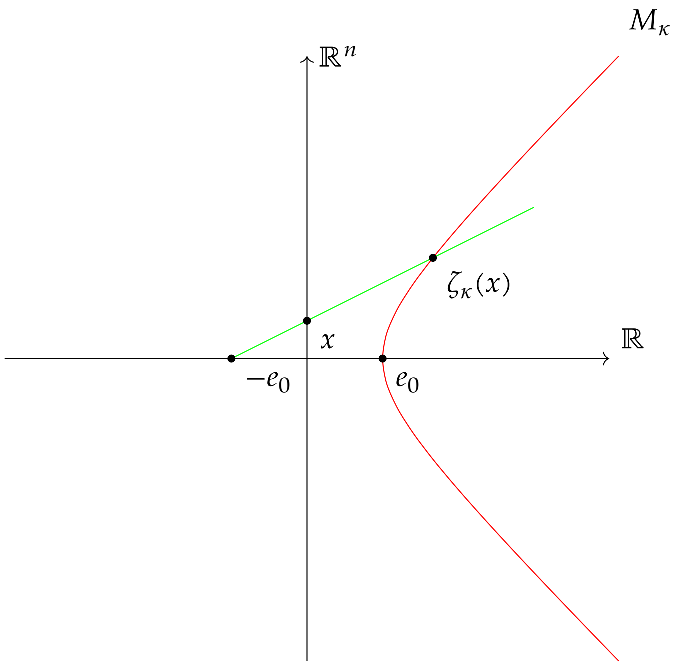


Figure 4.1: The projection of the point x.

4.2.1 *Formula for the Distance Function*

We now compute the distance function $d = d_\kappa^M$ of M_κ. Let $p \in M_\kappa$, with $d(p, e_0) = t$. Then $p = \gamma_v(t) = \mathrm{cs}_\kappa(t)e_0 + \mathrm{sn}_\kappa(t)v$ for a suitable $v \in S^{n-1}$. Thus we have $\mathrm{cs}_\kappa(d(e_0, p)) = \langle e_0, p\rangle_\kappa$. Since $\langle\cdot,\cdot\rangle_\kappa$ is invariant under $\mathrm{Iso}(M_\kappa)$ and $\mathrm{Iso}(M_\kappa)$ operates transitively, we have

$$\forall p, q \in M_k : \quad \mathrm{cs}_\kappa(d_\kappa^M(p,q)) = \langle p, q\rangle_\kappa. \tag{4.1}$$

4.2.1.1 *The Ball Model of M_κ*

We project the points of M_κ from $-e_0$ to $\mathbb{R}^n$. We have

$$\zeta_\kappa : \mathbb{R}^n \supset B_\kappa \to M_\kappa \subset \mathbb{R}^{1+n}$$

given by

$$\zeta_\kappa(x) = \left(\frac{1 - \kappa\|x\|^2}{1 + \kappa\|x\|^2}, \frac{2x}{1 + \kappa\|x\|^2} \right),$$

where

$$B_\kappa = \{x \in \mathbb{R}^n \mid 1 + \kappa\|x\|^2 > 0\} = \{x \mid \|x\| < \frac{1}{\sqrt{-\kappa}}\} \subset \mathbb{R}^n.$$

For $\kappa > 0$ the ball model describes all points of M_κ except $-e_0$ (which corresponds to the point at ∞ in this model).

We compute now the distance d^B_κ on B_κ such that

$$\zeta_\kappa : (B_\kappa, d^B_\kappa) \to (M_\kappa, d^M_\kappa)$$

is an isometry. For the computation let $d = d^B_\kappa$, then

$$\begin{aligned} \operatorname{cs}_\kappa(d(x,y)) &= \operatorname{cs}_\kappa(d^M_\kappa(\zeta_\kappa(x), \zeta_\kappa(y))) \\ &= \langle \zeta_\kappa(x), \zeta_\kappa(y) \rangle_\kappa \\ &= \frac{(1 - \kappa\|x\|^2)(1 - \kappa\|y\|^2) + 2\kappa\langle x, y \rangle}{(1 + \kappa\|x\|^2)(1 + \kappa\|y\|^2)} \\ &= 1 - \frac{2\kappa\|x - y\|^2}{(1 + \kappa\|x\|^2)(1 + \kappa\|y\|^2)}. \end{aligned}$$

Thus

$$\operatorname{cs}_\kappa(d^B(x,y)) = 1 - \frac{2\kappa\|x - y\|^2}{(1 + \kappa\|x\|^2)(1 + \kappa\|y\|^2)}. \tag{4.2}$$

This can be written in the following equivalent ways:

$$d(x,y) = \operatorname{arccs}_\kappa \left(1 - \frac{2\kappa\|x - y\|^2}{(1 + \kappa\|x\|^2)(1 + \kappa\|y\|^2)} \right) \tag{4.3}$$

$$= 2 \operatorname{arcsn}_\kappa \left(\frac{\|x - y\|}{\sqrt{1 + \kappa\|x\|^2}\sqrt{1 + \kappa\|y\|^2}} \right). \tag{4.4}$$

We now compute the euclidean distance out of d^B.

Proposition 20 (The case $\kappa = -1$ is Lemma 2.1 (A) from [LS07]). *Let $x, y \in B_\kappa$, $d = d^B_\kappa$ then*

$$\|x - y\| = \frac{\mathrm{sn}_\kappa(d(x,y)/2)}{\mathrm{cs}_\kappa(d(x,o)/2)\,\mathrm{cs}_\kappa(d(y,o)/2)}. \tag{4.5}$$

Proof. We use the formulas

$$\kappa\,\mathrm{sn}^2_\kappa(t/2) = \frac{1 - \mathrm{cs}_\kappa(t)}{2} \tag{a}$$

and

$$\mathrm{cs}^2_\kappa(t/2) = \frac{1 + \mathrm{cs}_\kappa(t)}{2} \tag{b}$$

then from Equation a together with Equation 4.2 we obtain:

$$\mathrm{sn}^2_\kappa(d(x,y)/2) = \frac{\|x - y\|^2}{(1 + \kappa\|x\|^2)(1 + \kappa\|y\|^2)}.$$

From Equation b those together with Equation 4.2 we obtain:

$$\mathrm{cs}^2_\kappa(d(x,o)/2) = 1 - \frac{\kappa\|x\|^2}{1 + \kappa\|x\|^2} = \frac{1}{1 + \kappa\|x\|^2}.$$

Combining those we obtain the statement. □

4.2.1.2 *Upper Half Space Model for $\kappa < 0$*

Now let $\kappa < 0$, let $\mathcal{U} \subset \mathbb{R}^n$ be the upper half space given by

$$\mathcal{U} := \{(x_1, \dots, x_n) \mid x_n > 0\}.$$

Let $\bar{\cdot} : \mathbb{R}^n \to \mathbb{R}^n$ be the conjugation given by

$$x = (x_1, \dots, x_n) \mapsto \bar{x} = (x_1, \dots, x_{n-1}, -x_n).$$

Let σ_κ be the involution at the sphere $S(\frac{1}{\sqrt{-\kappa}} e_n, \sqrt{-\frac{2}{\kappa}})$ with center $\frac{1}{\sqrt{-\kappa}} e_n$ and radius $\sqrt{-\frac{2}{\kappa}}$. Then $\sigma_\kappa(\overline{\mathcal{U}}) = B_\kappa$, where $\overline{\mathcal{U}}$ denotes the lower half space. Furthermore

$$\begin{aligned} \eta_\kappa : \quad & \mathcal{U} \longrightarrow B_\kappa \\ & x \longmapsto \sigma_\kappa(\bar{x}). \end{aligned}$$

On $\mathcal{U}$ we define the metric $d_\kappa^{\mathcal{U}}$ in a way such that η_κ is an isometry

$$(\mathcal{U}, d_\kappa^{\mathcal{U}} \to (B_\kappa, d_\kappa^B).$$

Thus

$$\begin{aligned} \mathrm{cs}_\kappa(d_\kappa^{\mathcal{U}}(x,y)) &= \mathrm{cs}_\kappa(d_\kappa^B(\eta_\kappa(x), \eta_\kappa(y))) \\ &= 1 - 2\frac{\kappa \|\eta_\kappa(x) - \eta_\kappa(y)\|^2}{(1 + \kappa\|\eta_\kappa(x)\|^2)(1 + \kappa\|\eta_\kappa(y)\|^2)}. \end{aligned}$$

Note that if $\sigma = \mathrm{Inv}_{S(a,r)}$ then

$$\|\sigma(\bar{x}) - \sigma(\bar{y})\| \doteq \frac{r^2 \|\bar{x} - \bar{y}\|}{\|\bar{x} - a\| \|\bar{y} - a\|}$$

by the definition for involutions. We therefore have

$$\eta_\kappa(x) = a + r^2 \frac{(\bar{x} - a)}{\|\bar{x} - a\|},$$

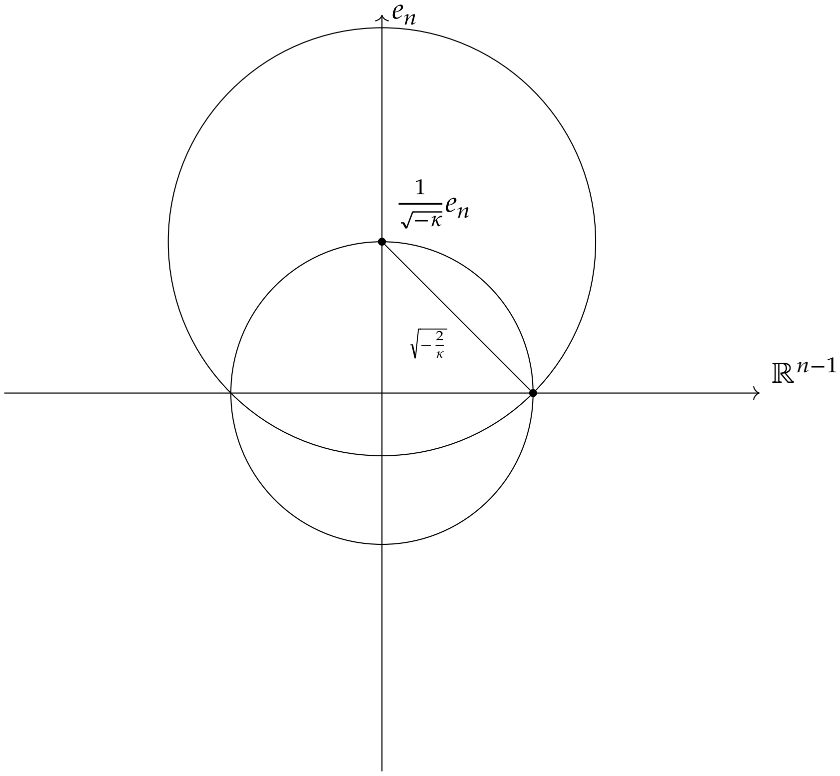


Figure 4.2: The upper half space model

where $a = \frac{1}{\sqrt{-\kappa}} e_n$ and $r = \sqrt{\frac{-2}{\kappa}}$. We thus compute

$$\begin{aligned} 1 + \kappa \|\eta_\kappa(x)\|^2 &= 1 + \kappa \left(\langle a, a \rangle - \frac{4}{\kappa} \frac{\langle \bar{x} - a, a \rangle}{\|\bar{x} - a\|^2} + \frac{4}{\kappa^2} \frac{1}{\|\bar{x} - a\|^2} \right) \\ &= -\frac{4}{\|\bar{x} - a\|^2} \left(\langle \bar{x} - a, a \rangle - \frac{1}{\kappa} \right) \\ &= -\frac{4}{\|\bar{x} - a\|^2} \langle \bar{x}, a \rangle. \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{\kappa \|\eta_\kappa(x) - \eta_\kappa(y)\|^2}{(1 + \kappa \|\eta_\kappa(x)\|^2)(1 + \kappa \|\eta_\kappa(y)\|^2)} \\ &= \frac{\|\bar{x} - a\|^2 \|y - a\|^2}{16 \langle \bar{x}, a \rangle \langle \bar{y}, a \rangle} \cdot \kappa \cdot \left(\frac{-2}{\kappa} \right)^2 \cdot \frac{\|\bar{x} - \bar{y}\|^2}{\|\bar{x} - a\|^2 \|\bar{y} - a\|^2} \\ &= \frac{\|\bar{x} - \bar{y}\|^2}{4\kappa \langle \bar{x}, a \rangle \langle \bar{y}, a \rangle} = -\frac{\|x - y\|^2}{4 x_n y_n}, \end{aligned}$$

thus

$$\mathrm{cs}_\kappa(d_\kappa^{\mathcal{U}}(x, y)) = 1 + \frac{\|x - y\|^2}{2 x_n y_n}. \tag{4.6}$$

And we get the formula

$$d_\kappa^{\mathcal{U}}(x, y) = \mathrm{arccs}_\kappa(1 + \frac{\|x - y\|^2}{2 x_n y_n})$$

for the distance function. This can also be written as follows:

$$\begin{aligned} d^{\mathcal{U}}(x, y) &= \mathrm{arccs}_\kappa \left(1 + \frac{\|x - y\|^2}{2 x_n y_n} \right) && (4.7) \\ &= 2 \,\mathrm{arcsn}_\kappa \left(\frac{1}{2\sqrt{-\kappa}} \sqrt{\frac{\|x - y\|^2}{2 x_n y_n}} \right) && (4.8) \\ &= 2 \ln \left(\frac{\|x - y\| + \|x - y + 2 y_n e_n\|}{2 \sqrt{x_n y_n}} \right) \frac{1}{\sqrt{-\kappa}}. && (4.9) \end{aligned}$$

VERTICAL GEODESICS IN $(\mathcal{U}, d_\kappa^{\mathcal{U}})$ We define

$$\mathrm{ex}_\kappa(t) := \exp(\sqrt{-\kappa}t)$$

and

$$\ln_\kappa(t) := \frac{\ln(t)}{\sqrt{-\kappa}}$$

(for $\kappa < 0$). Then $\gamma(t) := \frac{1}{\sqrt{-\kappa}} \mathrm{ex}_\kappa(t) e_n = \frac{1}{\sqrt{-\kappa}} \exp(\sqrt{-\kappa}t) \cdot e_n$ is the unit speed geodesic in $\mathcal{U}$ with $\gamma(0) = \frac{1}{\sqrt{-\kappa}} e_n$. Indeed:

$$\begin{aligned}
\mathrm{cs}_\kappa(d^{\mathcal{U}}(\gamma(t), \gamma(0))) &= 1 + \frac{(\mathrm{ex}_\kappa(t) - 1)^2}{2\,\mathrm{ex}_\kappa(t)} \\
&= \frac{1}{2}\left(\mathrm{ex}_\kappa(t) + \mathrm{ex}_\kappa(-t)\right) \\
&= \mathrm{cs}_\kappa(t) \\
\implies d_\kappa^{\mathcal{U}}(\gamma(t), \gamma(0)) &= t.
\end{aligned}$$

Now let B be the Busemann function of γ normalized such that $B(\gamma(t)) = -t$. Then

$$B(x) = -\frac{1}{\sqrt{-\kappa}} \ln(\sqrt{-\kappa} x_n).$$

We then compute:

Proposition 21 (The case $\kappa = -1$ is Lemma 2.1 (B) from [LS07]).

$$\|x - y\| = \frac{2\,\mathrm{sn}_\kappa\left(\frac{d_\kappa^{\mathcal{U}}(x,y)}{2}\right)}{\mathrm{ex}_\kappa\left(\frac{B(x)}{2}\right)\mathrm{ex}_\kappa\left(\frac{B(y)}{2}\right)}. \tag{4.10}$$

Proof. Using

$$\kappa\,\mathrm{sn}_\kappa^2(t/2) = \frac{1 - \mathrm{cs}_\kappa(t)}{2}$$

we have

$$-\kappa\,\mathrm{sn}_\kappa^2\left(\frac{d(x,y)}{2}\right) = \frac{\|x - y\|^2}{4 x_n y_n}$$

and
$$\mathrm{ex}_\kappa(B(x)/2) = \exp\left(-\frac{1}{2}\ln(\sqrt{-\kappa}x_n)\right) = \frac{1}{(\sqrt{-\kappa}x_n)^{\frac{1}{2}}}$$
thus
$$\mathrm{ex}_\kappa^2(B(x)/2)\,\mathrm{ex}_\kappa^2(B(y)/2) = \frac{1}{(-\kappa)x_n y_n}.$$

□

4.2.2 *Summary*

In the following we summarize the just computed model spaces. Rewriting the definitions from the previous chapter one can state the metric in several equivalent ways.

4.2.2.1 *Upper Half Space Model*

Let $\kappa < 0$ and $X = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$ and let

$$\begin{aligned} d(x,y) &= \operatorname{arccs}_\kappa\left(1 + \frac{\|x-y\|^2}{2x_n y_n}\right) \\ &= 2\operatorname{arcsn}_\kappa\left(\frac{1}{2\sqrt{-\kappa}}\sqrt{\frac{\|x-y\|^2}{x_n y_n}}\right) \\ &= 2\ln_\kappa\left(\frac{\|x-y\| + \|x-y+2y_n e_n\|}{2\sqrt{x_n y_n}}\right). \end{aligned}$$

And by Proposition 21 we get the corresponding formula to recover the euclidean distance by:

$$\|x-y\| = \frac{2\operatorname{sn}_\kappa\left(\frac{d_\kappa^U(x,y)}{2}\right)}{\operatorname{ex}_\kappa\left(\frac{B(x)}{2}\right)\operatorname{ex}_\kappa\left(\frac{B(y)}{2}\right)}.$$

4.2.2.2 *Disk Model*

Let $0 \neq \kappa \in \mathbb{R}$ and let $X = \{x \in \mathbb{R}^n \,|\, 1 + \kappa\|x\|^2 > 0\}$ with

$$\begin{aligned} d(x,y) &= \operatorname{arccs}_\kappa\left(1 - \frac{2\kappa\|x-y\|^2}{(1+\kappa\|x\|^2)(1+\kappa\|y\|^2)}\right) \\ &= 2\operatorname{arcsn}_\kappa\left(\frac{\|x-y\|}{\sqrt{1+\kappa\|x\|^2}\sqrt{1+\kappa\|y\|^2}}\right). \end{aligned}$$

By Proposition 20 we get can recover the euclidean distance with:

$$\|x-y\| = \frac{\operatorname{sn}_\kappa(d(x,y)/2)}{\operatorname{cs}_\kappa(d(x,o)/2)\operatorname{cs}_\kappa(d(y,o)/2)}.$$

For $\kappa = 0$ we just let $d(x, y) = 2\|x - y\|$ which is compatible with the second formula above.

Based on this observations we will define functions F and G in the next section which represent the formulas for recovering the euclidean distance.

4.3 RESULTS

4.3.1 *Spaces of Constant Sectional Curvature*

Definition 33. *In the following let*

$$F_\kappa(c; a, b) := \frac{\mathrm{sn}_\kappa(c/2)}{\mathrm{cs}_\kappa(a/2)\,\mathrm{cs}_\kappa(b/2)}$$

and

$$F(c; a, b) := F_{-1}(c; a, b)$$

furthermore let

$$G_\kappa(c; a, b) := \frac{2\sinh(c\sqrt{-\kappa}/2)}{\exp(a\sqrt{-\kappa}/2)\exp(b\sqrt{-\kappa}/2)} = \frac{2\,\mathrm{sn}_\kappa(c/2)\sqrt{|\kappa|}}{\exp(a\sqrt{|\kappa|}/2)\exp(b\sqrt{|\kappa|}/2)}.$$

We will prove the following result:

Theorem 12. *Let (X, d) be a simply connected, n-dimensional Riemannian manifold with constant sectional curvature $\kappa \in \mathbb{R}$. Let $o \in X$ be a base point and $\omega \in \partial_\infty X$ be some point on the geodesic boundary and let $\rho_o(x, y) = F_\kappa(d(x, y); d(x, o), d(y, o))$ and let $\rho_{\omega,o}(x, y) = G_\kappa(d(x, y); B(x), B(y))$ where B is the Busemann function of ω normalized such that $B(o) = 0$. Then:*

1. *If $\kappa = 0$, then the space (X, ρ_o) is a metric space isometric to the n-dimensional euclidean space.*
2. *If $\kappa < 0$, then the space (X, ρ_o) is a metric space isometric to the ball of radius $\frac{1}{\sqrt{-\kappa}}$ in n-dimensional euclidean space.*
3. *If $\kappa < 0$, then the space $(X, \rho_{\omega,o})$ is a metric space isometric to the upper half plane $\{x \in \mathbb{R}^n \mid x_0 > 0\}$.*
4. *If $\kappa > 0$, then the space (X, ρ_o) is isometric to the sphere $\mathbb{S}^n_r$ of radius $r = \frac{1}{\sqrt{\kappa}}$ with metric induced from the metric of $\mathbb{R}^{n+1}$.*

Collecting the calculations from the previous section, we can in each model space recover the euclidean distance using some calculation. We summarize this in the following proposition:

Lemma 2. 1. *Let x, y be two points in the Poincaré unit disk model space (X, d) (with curvature $\kappa < 0$) of the hyperbolic plane. Furthermore take $\omega \in X$ fixed. Denote by $a := d(x, \omega), b := d(y, \omega), c := d(x, y)$ the hyperbolic distances. Then the Euclidean distance between x and y is given by*

$$\|x - y\| = \frac{\mathrm{sn}_\kappa(c/2)}{\mathrm{cs}_\kappa(a/2)\,\mathrm{cs}_\kappa(b/2)} = \frac{\sinh(c\sqrt{-\kappa}/2)}{\cosh(a\sqrt{-\kappa}/2)\cosh(b\sqrt{-\kappa}/2)\sqrt{-\kappa}}.$$

2. *Let x, y be two points in the Poincaré upper half plane model space (with curvature $\kappa < 0$) of the hyperbolic plane. Let $\omega \in X$ be fixed. Let B be the Busemann function of ∞ normalized such that $B(o) = 0$. Denote by $a := B(x), b := B(y), c := d(x, y)$. Then the Euclidean distance between x and y is given by*

$$\|x - y\| = \frac{2\sinh(c\sqrt{-\kappa}/2)}{\exp(a\sqrt{-\kappa}/2)\exp(b\sqrt{-\kappa}/2)}.$$

3. *Let x, y, p be three points in the spherical model space (with curvature $\kappa > 0$). Denote $a := d(x, p), b := d(y, p), c := d(x, y)$. Then the Euclidean distance between x and y is given by*

$$\|x - y\| = \frac{\mathrm{sn}_\kappa(c/2)}{\mathrm{cs}_\kappa(a/2)\,\mathrm{cs}_\kappa(b/2)} = \frac{\sin(c\sqrt{\kappa}/2)}{\cos(a\sqrt{\kappa}/2)\cos(b\sqrt{\kappa}/2)\sqrt{\kappa}}.$$

Proof. This follows from the previous lemmas. Additional constructive proofs are given in the appendix. □

4.3.2 $CAT(\kappa)$-Spaces for $\kappa < 0$

Theorem 13. *Let $X = (X, d)$ be a complete $CAT(\kappa)$-space for $\kappa < 0$. Fix $o \in X$ and $\omega \in \partial_\infty X$. Then*

1. *The functions given by*

$$\rho_o(x, y) := F_\kappa(d(x, y); d(x, o), d(y, o))$$

and

$$\rho_{\omega,o}(x, y) := G_\kappa(d(x, y), B(x), B(y)),$$

where B is the Busemann function of ω normalized such that $B(o) = 0$, are metrics on X.

2. *We can extend ρ_o to a metric on $\bar{X} = X \cup \partial_\infty X$, and $\rho_{\omega,o}$ to a metric on $\bar{X} \setminus \{\omega\}$ and the following relations hold for $\xi, \eta \in \partial_\infty X \setminus \{\omega\}$:*

$$\rho_o(\xi, \eta) = \frac{2\exp(-(\xi|\eta)_o\sqrt{-\kappa})}{\sqrt{-\kappa}},$$

$$\rho_{\omega,o}(\xi, \eta) = e^{-(\xi|\eta)_{\omega,o}\sqrt{-\kappa}},$$

$$\rho_{\omega,o}(x, y) = \frac{2\rho_o(x, y)}{\rho_o(x, \omega)\rho_o(y, \omega)\sqrt{-\kappa}}.$$

3. *If $\gamma : (X, d) \to (X, d)$ is an isometry, then $\gamma : (X, \rho) \to (X, \rho)$ is a Möbius-map where $\rho = \rho_o$ or $\rho = \rho_{\omega,o}$.*

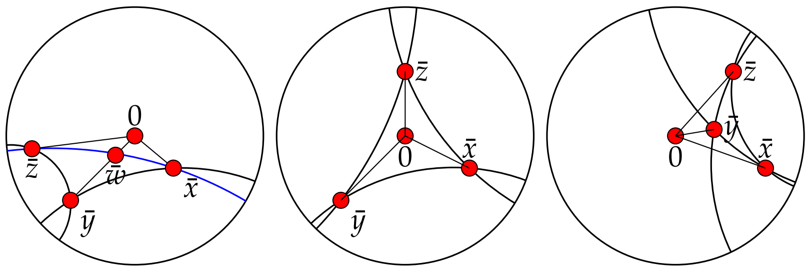
4. *The spaces (X, ρ_o) and $(X, \rho_{\omega,o})$ are Ptolemaic spaces.*

OUTLINE OF PROOF. In order to proof item 1, we have to show that the distance function ρ_o satisfies the triangle inequality. We use the fact that by the $\mathrm{CAT}(\kappa)$ inequality we can compare triangles in the model spaces. Furthermore we know that the triangle inequality holds when we apply the function F_κ to the metric in the model space. This means we only need to find suitable comparison triangles and then can apply the triangle inequality in the model space. If we find suitable inequalities to compare the distances in our space with the model distances, this allows us to show the claim. The construction we use works as follows. We embed two triangles (x, y, o) and (y, z, o) simultaneously in our comparison space. Denote the comparison triangles by $(\bar{x}, \bar{y}, 0)$ and $(\bar{y}, \bar{z}, 0)$ respectively. Furthermore denote by d the metric on the CAT(κ) space X, and by ρ_o the metric we get from applying the function F_κ. In the comparison space $(M^2_\kappa, \bar{d})$, we denote the metric by $\bar{d}$. Note that when applying the function F_κ to $\bar{d}$ we get the euclidean metric which we denote by $\| \cdot - \cdot \|$. We can therefore keep all the side lengths of those triangles unchanged while going to the comparison space. The only distance that can change is xz, i.e., $d(x, z)$ and $\bar{d}(x, z)$ may be different. If we require the triangles not to overlap in the comparison space, then there are three possible configurations in which the triangles may come to lay there. Those three cases are therefore analyzed separately. It is either possible to apply the CAT inequality to get the desired result, use a direct calculation or apply Alexandrov's Lemma to deform the triangle in order to get a suitable inequality. For the equalities in item 2 we have to extend the metric to the Gromov boundary. Note that in a CAT(κ) space the Gromov product on the boundary is well defined regardless of the sequence we choose. In particular $\lim(x_i, y_i)_o$ always exists for $x, y \in \partial_\infty X$. Therefore the Cauchy completion and the Gromov boundary coincide as sets. We can then use direct calculations to approximate the limit cases, while noting that for large $t \gg 0$, $\mathrm{sn}_\kappa(t)$ and $\mathrm{cs}_\kappa(t)$ behave approximately like $\exp(t)$. Item 3 is a simple direct computation. Item 4 follows by subembedding a 4-tuple of points in the comparison space. This is possible by the CAT(κ) condition. Then one can use the fact that the euclidean plane satisfies the Ptolemaic inequality to proof the claim for ρ_o. The equalities then also imply by use of item 4 that the triangle inequality holds for $\rho_{\omega,o}$.

Lemma 3 (Theorem 2.2 in [LS07]). *Let (X, d) be a complete CAT(κ)-space for $\kappa < 0$. Fix a base point $o \in X$. Then the function given by*

$$\rho_o(x, y) := F_\kappa(d(x, y); d(x, o), d(y, o)),$$

is a metric on X.



(a) Case 1 (b) Case 2 (c) Case 3

Figure 4.3: Cases for proof of Theorem 13 (1)

Proof. Let $x, y, z \in X$. We have to show that $\rho_o(x,z) \leq \rho_o(x,y) + \rho_o(y,z)$. Construct comparison triangles $\bar{x}, \bar{y}, 0$ and $\bar{y}, \bar{z}, 0$ in the (unit disk) model space $(\mathrm{MD}^2_\kappa, \bar{d})$. Choose the comparison triangle such that they do not overlap. The following cases (See Figure 4.3) are possible:

1. In the case the situation is like in figure (a). Then let $\bar{w}$ be the intersection point of the segments $[\bar{x}, \bar{z}]$ and $[0, \bar{y}]$ in the comparison space, and let $w \in [o, y]$ be the corresponding point with $d(w, o) = \bar{d}(\bar{w}, 0)$. We get:

$$\begin{aligned}
\rho_o(x,y) + \rho_o(y,z) &= F_\kappa(d(x,y); d(x,o), d(y,o)) + F_\kappa(d(y,z); d(y,o), d(z,o)) \\
&= F_\kappa(\bar{d}(\bar{x},\bar{y}); \bar{d}(\bar{x},0), \bar{d}(\bar{y},0)) + F_\kappa(\bar{d}(\bar{y},\bar{z}); \bar{d}(\bar{y},0), \bar{d}(\bar{z},0)) \\
&= \|\bar{x} - \bar{y}\| + \|\bar{y} - \bar{z}\| \\
&\geq \|\bar{x} - \bar{z}\| \\
&= F_\kappa(\bar{d}(\bar{x},\bar{z}); \bar{d}(\bar{x},0), \bar{d}(\bar{z},0)) \\
&= F_\kappa(\bar{d}(\bar{x},\bar{w}) + \bar{d}(\bar{w},\bar{z}); \bar{d}(\bar{x},0), \bar{d}(\bar{z},0)) \\
&\geq F_\kappa(d(x,w) + d(w,z); d(x,o), d(z,o)) \\
&\geq F_\kappa(d(x,z); d(x,o), d(z,o)) \\
&= \rho_o(x,y).
\end{aligned}$$

2. In the situation like in figure (b) we calculate:

$$\begin{aligned}
\rho_o(x,y) + \rho_o(y,z) &= F_\kappa(d(x,y); d(x,o), d(y,o)) + F_\kappa(d(y,z); d(y,o), d(z,o)) \\
&= F_\kappa(\bar{d}(\bar{x},\bar{y}); \bar{d}(\bar{x},0), \bar{d}(\bar{y},0)) + F_\kappa(\bar{d}(\bar{y},\bar{z}); \bar{d}(\bar{y},0), \bar{d}(\bar{z},0)) \\
&= \|\bar{x} - \bar{y}\| + \|\bar{y} - \bar{z}\| \\
&\geq \|\bar{x} - 0\| + \|\bar{z} - 0\| \\
&= F_\kappa(\bar{d}(\bar{x},0); \bar{d}(\bar{x},0), \bar{d}(0,0)) + F_\kappa(\bar{d}(\bar{z},0); \bar{d}(\bar{z},0), \bar{d}(0,0)) \\
&= F_\kappa(d(x,o); d(x,o), d(o,o)) + F_\kappa(d(z,o); d(z,o), d(o,o)) \\
&= \frac{\mathrm{sn}_\kappa(d(x,o)/2)}{\mathrm{cs}_\kappa(d(x,o)/2)} + \frac{\mathrm{sn}_\kappa(d(z,o)/2)}{\mathrm{cs}_\kappa(d(z,o)/2)} \\
&= \frac{\mathrm{sn}_\kappa(d(x,o)/2)\,\mathrm{cs}_\kappa(d(z,o)/2) + \mathrm{sn}_\kappa(d(z,o)/2)\,\mathrm{cs}_\kappa(d(x,o)/2)}{\mathrm{cs}_\kappa(d(z,o)/2)\,\mathrm{cs}_\kappa(d(x,o)/2)} \\
&= \frac{\mathrm{sn}_\kappa(d(x,o)/2 + d(z,o)/2)}{\mathrm{cs}_\kappa(d(z,o)/2)\,\mathrm{cs}_\kappa(d(x,o)/2)} \\
&= F_\kappa(d(x,o) + d(z,o); d(x,o), d(z,o)) \\
&\geq F_\kappa(d(x,z); d(x,o), d(z,o)) \\
&= \rho_o(x,z).
\end{aligned}$$

3. In case (c) we deform the non-convex polygon $0, \bar{x}, \bar{y}, \bar{z}$ to a convex polygon $0, \bar{x}', \bar{y}', \bar{z}'$ with the same side lengths and such that

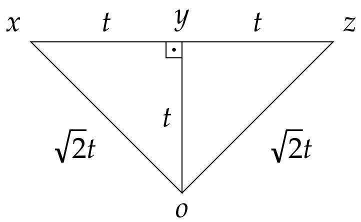


Figure 4.4: Triangle for counterexample

$\bar{y}'$ lies on the hyperbolic segment $[\bar{x}', \bar{z}']$. By Alexandrov's lemma (see 2.16 in [BH99]) $\bar{d}(\bar{y}', 0) \geq \bar{d}(\bar{y}, 0)$, therefore:

$$\begin{aligned}
\rho_o(x,y) + \rho_o(y,z) &= F_\kappa(d(x,y); d(x,o), d(y,o)) + F_\kappa(d(y,z); d(y,o), d(z,o)) \\
&= F_\kappa(\bar{d}(\bar{x},\bar{y}); \bar{d}(\bar{x},0), \bar{d}(\bar{y},0)) + F_\kappa(\bar{d}(\bar{y},\bar{z}); \bar{d}(\bar{y},0), \bar{d}(\bar{z},0)) \\
&\geq F_\kappa(\bar{d}(\bar{x}',\bar{y}'); \bar{d}(\bar{x}',0), \bar{d}(\bar{y}',0)) + F_\kappa(\bar{d}(\bar{y}',\bar{z}'); \bar{d}(\bar{y}',0), \bar{d}(\bar{z}',0)) \\
&= \|\bar{x}' - \bar{y}'\| + \|\bar{y}' - \bar{z}'\| \\
&\geq \|\bar{x}' - \bar{z}'\| \\
&= F_\kappa(\bar{d}(\bar{x}',\bar{z}'); \bar{d}(\bar{x}',0), \bar{d}(\bar{z}',0)) \\
&= F_\kappa(\bar{d}(\bar{x}',\bar{y}') + \bar{d}(\bar{y}',\bar{z}'); \bar{d}(\bar{x}',0), \bar{d}(\bar{z}',0)) \\
&= F_\kappa(d(x,y) + d(y,z); d(x,o), d(z,o)) \\
&\geq F_\kappa(d(x,z); d(x,o), d(z,o)) \\
&= \rho_o(x,z).
\end{aligned}$$

□

Remark 14. *One would think that for $\kappa > 0$ one could get a similar result by replacing the above formula in Theorem 13 with the spherical version. However this is not the case as the following counterexample shows: Consider a triangle in the plane $\mathbb{R}^2$ as shown in Figure 4.4. We can calculate:*

$$\rho_o(x,z) = \frac{\sin\left(\frac{2t}{2}\right)}{\cos\left(\frac{\sqrt{2}t}{2}\right)^2} = \frac{\sin(t)}{\cos\left(\frac{\sqrt{2}t}{2}\right)^2}$$

and

$$\rho_o(x,y) + \rho_o(y,z) = 2\frac{\sin\left(\frac{t}{2}\right)}{\cos\left(\frac{\sqrt{2}t}{2}\right)\cos\left(\frac{t}{2}\right)}.$$

Let

$$f(t) := \rho_o(x,z) - \rho_o(x,y) - \rho_o(y,z) = \frac{\sin(t)}{\cos\left(\frac{\sqrt{2}t}{t}\right)^2} - \frac{2\sin\left(\frac{t}{2}\right)}{\cos\left(\frac{\sqrt{2}t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$= \frac{\sin(t)\cos\left(\frac{t}{2}\right) - 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{\sqrt{2}t}{2}\right)}{\cos\left(\frac{\sqrt{2}t}{2}\right)^2\cos\left(\frac{t}{2}\right)}$$

$$= \sec\left(\frac{t}{\sqrt{2}}\right)\left(\sin(t)\sec\left(\frac{t}{\sqrt{2}}\right) - 2\tan\left(\frac{t}{2}\right)\right).$$

It is easy to see that $f(0) = 0$ and that the Taylor series expansion at $t = 0$ is:

$$\frac{t^5}{96} + \frac{t^7}{180} + \frac{181t^9}{92160} + \frac{5633t^{11}}{9676800} + O(t^{13}).$$

In particular this function is positive for small positive t. This is a contradiction to the triangle inequality. Therefore the approach as above does not work in the spherical case. However we will later see that a similar theorem holds for CBB-spaces.

Lemma 4 (The proof follows the sketch in [LS07])**.** *Let (X,d) be a CAT(κ)-space for $\kappa < 0$, $o \in X$ a base point. And let $\rho_o(x,y) = F_\kappa(d(x,y); d(p,x), d(p,y))$. Then ρ_o satisfies the Ptolemaic inequality for any $x_1, x_2, x_3, x_4 \in X$:*

$$\rho_o(x_1,x_3)\rho_o(x_2,x_4) \le \rho_o(x_1,x_2)\rho_o(x_3,x_4) + \rho_o(x_1,x_4)\rho_o(x_2,x_3).$$

Proof. X satisfies the CAT(κ) 4-point condition (Proposition 1.11, p. 164 in [BH99]). This means for any 4-tuple of points (x_1, x_2, x_3, x_4) in X there exists a 4-tuple of comparison points $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ in $(M^2_\kappa, \bar{d})$ such that $d(x_1,x_2) = \bar{d}(\bar{x}_1,\bar{x}_2), d(x_2,x_3) = \bar{d}(\bar{x}_2,\bar{x}_3), d(x_3,x_4) = \bar{d}(\bar{x}_3,\bar{x}_4)$,

$d(x_4, x_1) = \bar{d}(\bar{x}_4, \bar{x}_1)$. And furthermore $d(x_1, x_3) \leq \bar{d}(\bar{x}_1, \bar{x}_3)$ and $d(x_2, x_4) \leq \bar{d}(\bar{x}_2, \bar{x}_4)$. Note that $F_\kappa(\bar{d}(\bar{x}, \bar{y}), \bar{d}(\bar{x}, \bar{o}), \bar{d}(\bar{y}, \bar{o})) = \|x - y\|$. Furthermore we know that the Euclidean metric satisfies the Ptolemaic inequality:

$$\|\bar{x}_1 - \bar{x}_3\| \cdot \|\bar{x}_2 - \bar{x}_4\| \leq \|\bar{x}_1 - \bar{x}_2\| \cdot \|\bar{x}_3 - \bar{x}_4\| + \|\bar{x}_1 - \bar{x}_4\| \cdot \|\bar{x}_2 - \bar{x}_3\|$$

This implies that:

$$\begin{aligned} &\operatorname{sn}_\kappa(d(x_1, x_3)/2) \operatorname{sn}_\kappa(d(x_2, x_4)/2) \\ &\qquad \leq \operatorname{sn}_\kappa(\bar{d}(\bar{x}_1, \bar{x}_3)/2) \operatorname{sn}_\kappa(\bar{d}(\bar{x}_2, \bar{x}_4)/2) \\ &\qquad \leq \operatorname{sn}_\kappa(\bar{d}(\bar{x}_1, \bar{x}_2)/2) \operatorname{sn}_\kappa(\bar{d}(\bar{x}_3, \bar{x}_4)/2) \\ &\qquad\quad + \operatorname{sn}_\kappa(\bar{d}(\bar{x}_1, \bar{x}_4)/2) \operatorname{sn}_\kappa(\bar{d}(\bar{x}_2, \bar{x}_3)/2) \\ &\qquad = \operatorname{sn}_\kappa(d(x_1, x_2)/2) \operatorname{sn}_\kappa(d(x_3, x_4)/2) \\ &\qquad\qquad + \operatorname{sn}_\kappa(d(x_1, x_4)/2) \operatorname{sn}_\kappa(d(x_2, x_3)/2). \end{aligned}$$

From this the claim follows. □

Remark 15. *Note that this still holds if one of the points is on the Gromov boundary. Let (X, d) be a CAT(κ)-space for $\kappa < 0$ with base point $o \in X$. And let $\rho_o(x, y) = F_\kappa(d(x, y); d(p, x), d(p, y))$. Then ρ_o satisfies the Ptolemaic inequality for any $x_1, x_2, x_3 \in X$, $\omega \in \partial_\infty X$:*

$$\rho_o(x_1, x_3)\rho_o(x_2, \omega) \leq \rho_o(x_1, x_2)\rho_o(x_3, \omega) + \rho_o(x_1, \omega)\rho_o(x_2, x_3).$$

This holds because we have

$$\begin{aligned} &\operatorname{sn}_\kappa (d(x_1, x_3)/2) \operatorname{sn}_\kappa (d(x_2, \omega_i)/2) \\ &\leq \operatorname{sn}_\kappa (d(x_1, x_2)/2) \operatorname{sn}_\kappa (d(x_3, \omega_i)/2) + \operatorname{sn}_\kappa (d(x_1, \omega_i)/2) \operatorname{sn}_\kappa (d(x_2, x_3)/2), \end{aligned}$$

for any $i \in \mathbb{N}$, in particular it holds in the limit as well.

Lemma 5 (For the case $\kappa = -1$, the points 1.-3. and 5. are from Theorem 1.1 (C-E) in [LS07])**.** *Let (X, d) be a CAT(κ)-space for $\kappa < 0$ and let $\partial_\infty X$ denote the boundary at infinity. Then we can extend the metrics $\rho_o(x, y) = F_\kappa(d(x, y); d(p, x), d(p, y))$ and $\rho_{\omega,o}(x, y) = G_\kappa(d(x, y), B(x), B(y))$ (here B is the Busemann function of ω normalized such that $B(o) = 0$) to the boundary at infinity and the following holds for $\xi, \eta \in \partial_\infty X \setminus \{\omega\}$:*

1.
$$\rho_{\omega,o}(\xi,\eta) = e^{-(\xi|\eta)_{\omega,o}\sqrt{-\kappa}},$$

2.
$$\rho_o(\xi,\eta) = \frac{2\exp(-(\xi|\eta)_o\sqrt{-\kappa})}{\sqrt{-\kappa}},$$

3.
$$\rho_{\omega,o}(x,y) = \frac{2\rho_o(x,y)}{\rho_o(x,\omega)\rho_o(y,\omega)\sqrt{-\kappa}},$$

4.
$$\rho_o(o,\eta) = \frac{1}{\sqrt{-\kappa}}$$

5. *If $\gamma : (X,d) \to (X,d)$ is an isometry, then $\gamma : (X,\rho) \to (X,\rho)$ is a Möbius-map where $\rho = \rho_o$ or $\rho = \rho_{\omega,o}$.*

Proof. Define $\rho_o(x,\eta)$ as the limit of some sequence $\{\eta_i\}_i \subset X$ converging to η:
$$\rho_o(x,\eta) = \lim_{i\to\infty} \rho_o(x,\eta_i)$$
define furthermore
$$\rho_o(\xi,\eta) = \lim_{i\to\infty} \rho_o(\xi_i,\eta_i).$$
And analogously for $\rho_{\omega,o}$. By a result of Bourdon ([Bou96] and Exercise 3.18 in [BH99]) those definitions do not depend on the choice of sequences and the limit always exists (also see [FS11]). We have:
$$\begin{aligned}
\rho_o(x,\eta) &= \lim_{i\to\infty} \frac{\mathrm{sn}_\kappa(d(x,\eta_i)/2)}{\mathrm{cs}_\kappa(d(x,o)/2)\,\mathrm{cs}_\kappa(d(\eta_i,o)/2)} \\
&= \lim_{i\to\infty} \frac{\exp(d(x,\eta_i)\sqrt{-\kappa}/2)}{\mathrm{cs}_\kappa(d(x,o)/2)\exp(d(\eta_i,o)\sqrt{-\kappa}/2)\sqrt{-\kappa}} \\
&= \lim_{i\to\infty} \frac{\exp(\frac{\sqrt{-\kappa}}{2}(d(x,\eta_i)-d(\eta_i,o)))}{\mathrm{cs}_\kappa(d(x,o)/2)\sqrt{-\kappa}} \\
&= \frac{\exp(\frac{\sqrt{-\kappa}}{2}B_{\omega,o}(x))}{\mathrm{cs}_\kappa(d(x,o)/2)\sqrt{-\kappa}}.
\end{aligned}$$

From this it follows that

$$\begin{aligned}\rho_{\omega,o}(x,y) &= \frac{2\,\mathrm{sn}_\kappa(d(x,y)/2)\sqrt{-\kappa}}{\exp(\sqrt{-\kappa}B_{\omega,o}(x)/2)\exp(\sqrt{-\kappa}B_{\omega,o}(y)/2)}\\ &= \frac{2\rho_o(x,y)}{\rho_o(x,\omega)\rho_o(y,\omega)\sqrt{-\kappa}}.\end{aligned}$$

For $\xi \neq \eta$ we have

$$\begin{aligned}\rho_o(\xi,\eta) &= \lim_{i\to\infty}\frac{\mathrm{sn}_\kappa(d(\xi_i,\eta_i)/2)}{\mathrm{cs}_\kappa(d(\xi_i,o)/2)\,\mathrm{cs}_\kappa(d(\eta_i,o)/2)}\\ &= \lim_{i\to\infty}\frac{2\exp(d(\xi_i,\eta_i)\sqrt{-\kappa}/2)}{\exp(d(\xi_i,o)\sqrt{-\kappa}/2)\exp(d(\eta_i,o)\sqrt{-\kappa}/2)\sqrt{-\kappa}}\\ &= \lim_{i\to\infty}2\exp(\frac{\sqrt{-\kappa}}{2}(d(\xi_i,\eta_i)-d(\xi_i,o)-d(\eta_i,o)))\frac{1}{\sqrt{-\kappa}}\\ &= \lim_{i\to\infty}2\exp(-\sqrt{-\kappa}(\xi_i|\eta_i)_o)\frac{1}{\sqrt{-\kappa}}\\ &= \frac{2e^{-(\xi|\eta)_o\sqrt{-\kappa}}}{\sqrt{-\kappa}}.\end{aligned}$$

We furthermore get that for $\xi,\eta \in \partial_\infty X \setminus \{\omega\}$:

$$\begin{aligned}\rho_{\omega,o}(\xi,\eta) &= \frac{2\rho_o(\xi,\eta)}{\rho_o(\xi,\omega)\rho_o(\eta,\omega)\sqrt{-\kappa}}\\ &= \frac{e^{-(\xi|\eta)_o\cdot\sqrt{-\kappa}}}{e^{-(\xi|\omega)_o\cdot\sqrt{-\kappa}}\cdot e^{-(\eta|\omega)_o\cdot\sqrt{-\kappa}}}\\ &= e^{\sqrt{-\kappa}(-(\xi|\eta)_o+(\xi|\omega)_o+(\eta|\omega)_o)}\\ &= e^{-(\xi|\eta)_{\omega,o}\sqrt{-\kappa}}.\end{aligned}$$

Let $\eta \in \partial_\infty X$ then we have[2]

$$\rho_o(o,\eta) = \lim_{i\to\infty} \frac{\mathrm{sn}_\kappa(d(o,\eta_i)/2)}{\mathrm{cs}_\kappa(d(o,o)/2)\,\mathrm{cs}_\kappa(d(o,\eta_i)/2)} \tag{4.11}$$

$$= \lim_{i\to\infty} \frac{\mathrm{sn}_\kappa(d(o,\eta_i)/2)}{\mathrm{cs}_\kappa(d(o,\eta_i)/2)} \tag{4.12}$$

$$= \lim_{i\to\infty} \frac{\sinh(\sqrt{-\kappa}d(o,\eta_i)/2)}{\cosh(\sqrt{-\kappa}d(o,\eta_i)/2)\sqrt{-\kappa}} \tag{4.13}$$

$$= \frac{1}{\sqrt{-\kappa}}. \tag{4.14}$$

Similarly we calculate

$$\rho_{\omega,o}(o,\eta) = \lim_{i\to\infty} \frac{2\,\mathrm{sn}_\kappa(d(o,\eta_i)/2)}{\exp(\sqrt{-\kappa}B_{\omega,o}(o))\exp(\sqrt{-\kappa}B_{\omega,o}(\eta_i))}$$
$$= \lim_{i\to\infty} \frac{2\sinh(\sqrt{-\kappa}d(o,\eta_i)/2)}{\sqrt{-\kappa}\exp(B_{\omega,o}(\eta_i))}.$$

Given a isometry $\gamma : (X,d) \to (X,d)$ we have:

$$\mathrm{cr}((x,y,z,w),\rho_o) = \frac{\rho_o(x,z)\rho_o(y,w)}{\rho_o(x,y)\rho_o(z,w)} = \frac{\mathrm{sn}_\kappa(d(x,z)/2)\,\mathrm{sn}_\kappa(d(y,w)/2)}{\mathrm{sn}_\kappa(d(x,y)/2)\,\mathrm{sn}_\kappa(d(z,w)/2)}.$$

In particular, the $\mathrm{cs}_\kappa(d(x,o)/2)$ terms all cancel out and it follows that

$$\mathrm{cr}((x,y,z,w),\rho_o) = \mathrm{cr}((\gamma(x),\gamma(y),\gamma(z),\gamma(w)),\rho_o).$$

□

2 Note that this does not depend on the sequence chosen regardless of the properties of the space as long as $\eta_i \to \infty (i \to \infty)$.

4.3.3 $CBB(\kappa)$-Spaces for $\kappa > 0$

For spaces that have curvature bounded from below in the sense of Alexandrov transforming the metric using the transformation given by the function F still results in a new metric. In particular we have the following result.

Theorem 14. *Let (X, d) be a complete intrinsic $CBB(\kappa)$-space with $\kappa > 0$, then*

$$\rho_o(x, y) = F_\kappa(d(x, y); d(x, o), d(y, o))$$

is a metric.

This result answers a question[3] originally asked by Marc Lischka.

The proof uses the following version of the Kirszbraun theorem which can be found in [LS97] and [AKP10]:

Lemma 6 (Kirszbraun theorem)**.** *Let (X, d) be a complete intrinsic space. Then (X, d) is $CBB(\kappa)$ if and only if for any 3-point set $V_3 \subset X$ and any 4-point set $V_4 \supset V_3$ in X, any short map $f : V_3 \to M^2_\kappa$ can be extended to a short map $F : V_4 \to M^2_\kappa$ (so that $f = F|_{V_3}$).*

We also need a bound on the diameter of X which follows from the CBB property.

Theorem 15 (Theorem 10.4.1 in [BBI01])**.** *Let (X, d) be a complete intrinsic $CBB(\kappa)$ space with $\kappa > 0$, then* $\mathrm{diam}(X) \leq \frac{\pi}{\sqrt{\kappa}}$.

Proof of Theorem 14. Let $x, y, z, o \in X$. We want to show that $\rho_o(x, z) \leq \rho_o(x, y) + \rho_o(y, z)$. Take the comparison triangle $(\bar{x}, \bar{z}, \bar{o})$ in the model space $(M^2_\kappa, \bar{d})$. By the above lemma we can find a comparison point $\bar{y}$ such that $d(x, y) \geq \bar{d}(\bar{x}, \bar{y})$, $d(z, y) \geq \bar{d}(\bar{z}, \bar{y})$ and $d(o, y) \geq \bar{d}(\bar{o}, \bar{y})$ and for the other distances we have equality. By Proposition 20 we know

3 Question: For which combinations of κ and κ' is

$$\rho_o(x, y) = F_\kappa(d(x, y); d(x, o), d(y, o))$$

a metric on $M^n_{\kappa'}$?

that the triangle inequality holds for ρ_o in the model space, therefore we have:

$$\begin{aligned}
\rho_o(x,z) &= \bar{\rho}_o(\bar{x},\bar{z}) \\
&\leq \bar{\rho}_o(\bar{x},\bar{y}) + \bar{\rho}_o(\bar{y},\bar{z}) \\
&= \frac{\operatorname{sn}_\kappa\left(\frac{\bar{d}(\bar{x},\bar{y})}{2}\right)}{\operatorname{cs}_\kappa\left(\frac{\bar{d}(\bar{x},\bar{o})}{2}\right)\operatorname{cs}_\kappa\left(\frac{\bar{d}(\bar{y},\bar{o})}{2}\right)} + \frac{\operatorname{sn}_\kappa\left(\frac{\bar{d}(\bar{y},\bar{z})}{2}\right)}{\operatorname{cs}_\kappa\left(\frac{\bar{d}(\bar{y},\bar{o})}{2}\right)\operatorname{cs}_\kappa\left(\frac{\bar{d}(\bar{z},\bar{o})}{2}\right)} \\
&\leq \frac{\operatorname{sn}_\kappa\left(\frac{d(x,y)}{2}\right)}{\operatorname{cs}_\kappa\left(\frac{d(x,o)}{2}\right)\operatorname{cs}_\kappa\left(\frac{d(y,o)}{2}\right)} + \frac{\operatorname{sn}_\kappa\left(\frac{d(y,z)}{2}\right)}{\operatorname{cs}_\kappa\left(\frac{d(y,o)}{2}\right)\operatorname{cs}_\kappa\left(\frac{d(z,o)}{2}\right)} \\
&\leq \rho_o(x,y) + \rho_o(y,z).
\end{aligned}$$

In the second inequality we used the bound on the diameter of X. $\square$

4.3.4 *δ-Hyperbolic Spaces*

Let (X, d) be a metric space. For $o \in X$ define[4]:

$$\rho(x,y) := F_{-1}(d(x,y); d(x,o), d(y,o)) = \frac{\sinh(d(x,y)/2)}{\cosh(d(o,x)/2)\cosh(d(o,y)/2)}.$$

For an arbitrary metric space (X, d), ρ is not generally a metric. However, one can apply a chain construction to try to get a metric $\bar{\rho}$ out of ρ. Let $\bar{\rho}$ be the metric[5] constructed as follows

$$\bar{\rho}(x,y) := \inf \sum_{i=0}^{n-1} \rho(x_i, x_{i+1}),$$

where the infimum runs over all chains of finite length $x = x_0, \ldots, x_n = y$ of points in X. In case that (X, d) is a δ-hyperbolic metric space with $\delta < \ln(2)$ we get that $(X, \bar{\rho})$ is a metric space and ρ is bi-Lipschitz to $\bar{\rho}$. Furthermore we can Cauchy complete the metric and extend it to $\bar{X} = X \cup \partial X$. By abuse of notation we also write $\bar{\rho}$ for the completed metric.

Theorem 16. *Let (X, d) be a δ-hyperbolic metric space with $0 \leq \delta < \ln(2)$. Then $\bar{\rho}$ is a metric and $\bar{\rho} \leq \rho \leq \lambda\bar{\rho}$ for some $\lambda \geq 1$. Furthermore (for any δ) we have $\omega \in \partial X$ if and only if $\bar{\rho}(o, \omega) = 1$. And as sets $\partial_\infty X = \partial X$.*

Remark 16. *This result does not hold for δ-hyperbolic spaces with $\delta > \ln(2)$. In particular, in the appendix we construct an example of a δ-hyperbolic metric space with $\delta > \ln(2)$ where the metrics ρ and $\bar{\rho}$ can not be bi-Lipschitz to each other. See Section A.3.*

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4 We drop the specification of the base point from the metric whenever it is clear from the context. Also in this section generally $\kappa = -1$.

5 Metrizing a semimetric (metric without triangle inequality) or premetric (semimetric where $d(x,y) = 0$ for $x \neq y$ is allowed) using this construction always results in at least a pseudo-metric (triangle inequality but $d(x,y) = 0$ allowed). In this case however we always get a metric. As we will see later in Lemma 13 and Theorem 17.

OUTLINE OF PROOF. The proof of Theorem 16 is quite involved. We give a short overview of all the elements and parts of the proof. One can show that in a δ-hyperbolic metric space (X, d) with $\delta < \ln(2)$ there exists some constant $s_0 > 0$ such that whenever the points $x, y, z, o \in X$ have pairwise distance greater than s_0, then the largest two members m_0, m_1 of the triple $(\rho(x, y), \rho(y, z), \rho(z, x))$ satisfy $\frac{1}{2} \leq \frac{m_0}{m_1} \leq 2$ (Lemma 8). Such a triple is said to satisfy the 2-quasi-metric inequality. In turn by a result of Frink (Lemma 9) points for which each triple satisfies the above inequality can be approximated as follows: Given a chain of points $x = x_0, \dots, x_n = y$ in a semimetric space (X, d) (the space may not satisfy the triangle inequality) X such that all points satisfy the 2-quasi-metric inequality, one can calculate the distance between x and y as:

$$\frac{1}{4} d(x, y) \leq \sum_{i=0}^{n-1} d(x_i, x_{i+1}).$$

In particular in a δ-hyperbolic space with $\delta < \ln(2)$, this allows one to approximate the distances in the ρ semimetric given the condition that all involved points have pairwise distances (and distances to o), greater than s_0 in the metric d. For points $x, y \in X$ that are close together in the metric d (i.e., $d(x, y) \leq s$), it turns out that regardless of the metric space (X, d), there exists some constant $\mu_1(s)$ depending only on s such that we can approximate $\bar{\rho}$ by $\bar{\rho}(x, y) \geq \mu_1(s)\rho(x, y)$ (Lemma 13). A similar result can be constructed for points $x, y, z \in X$ such that $d(x, y) \leq s$ and $d(x, z) > s$, in this case there also exists a constant $\mu_2(s) > 0$ and we get the following inequality (Lemma 15):

$$\mu_2(s)\rho(x, z) \leq \rho(x, y) + \rho(y, z).$$

For points that have a large radial part (the distance from the base point), we can approximate ρ by ρ_{rad} which is the pseudo-metric resulting from constructing ρ using $d_{\text{rad}}(x, y) = |d(x, o) - d(y, o)|$ instead of d. Because this satisfies (Lemma 10) the triangle inequality, this allows a direct calculation in those cases. The idea of the proof of Theorem 16 is now as follows: Let (X, d) be a δ-hyperbolic metric space (X, d) with

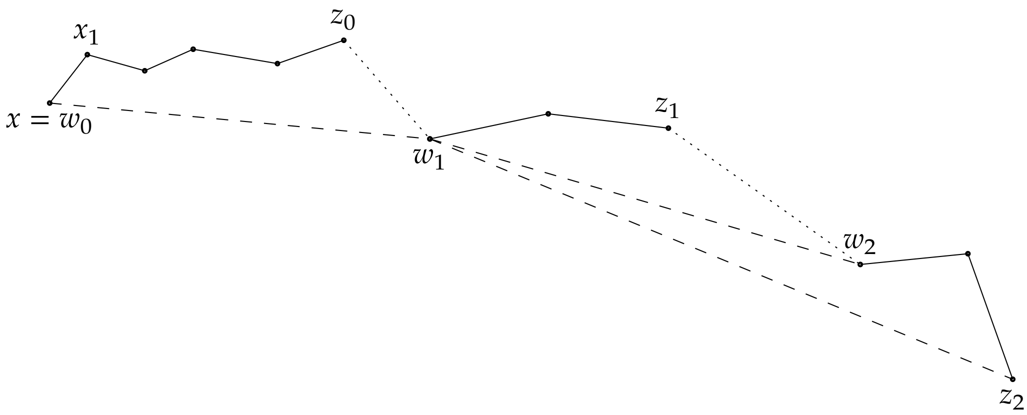

Figure 4.5: Subdivision of chain

$\delta < \ln(2)$ a base point $o \in X$ and a chain $x = x_0, \dots, x_n = y$. If there is at least one point of the chain close to the base point $(d(o, x_k) \leq s_0)$ we can approximate by either Lemma 13 or using the radial form of the metric. In the other case we know that $d(o, x_i) > s_0$ for any index i. We then subdivide our chain in such a way that we have sub-chains of length $\leq s_0$ and we choose the sub-indices such that we take the maximal possible indices (e.g.,if we end the sub-chain at any later index it is longer than s_0). See also Figure 4.5. We label the points beginning and ending such sub-chains with w_i and z_i. We then have $d(w_i, z_i) \leq s_0$ and $d(w_i, w_j) > s_0$ for $i \neq j$. We can then use a combination of the results (Lemma 13 and Lemma 15) to simplify the whole chain such that all remaining points have distance greater than s_0. Then one can apply Frink's result (Lemma 9) to get $\bar{\rho}(x, y) \geq \lambda\rho(x, y)$, which completes the proof.

Lemma 7. *For every $C > 0$ there exists a $c_0 > 0$ such that for $a, b, c, d > c_0$ whenever $\sinh(a)\cosh(b) \geq \sinh(c)\cosh(d)$ then $a + b + C \geq c + d$.*

Proof. We calculate

$$\begin{aligned}\sinh(a)\cosh(b) &= \frac{1}{4}(\exp(a)-\exp(-a))(\exp(b)+\exp(-b))\\ &= \frac{1}{4}\exp(a+b)\left(1+\exp(-2b)-\exp(-2a)-\exp(-2a-2b)\right)\\ &\leq \frac{1}{4}\exp(a+b)\left(1+\exp(-2b)\right)\\ &\leq \frac{1}{4}\exp(a+b)\left(1+\exp(-2c_0)\right).\end{aligned}$$

Furthermore

$$\begin{aligned}\sinh(c)\cosh(d) &= \frac{1}{4}(\exp(c)-\exp(-d))(\exp(c)+\exp(-d))\\ &= \frac{1}{4}\exp(c+d)\left(1+\exp(-2d)-\exp(-2c)-\exp(-2c-2d)\right)\\ &\geq \frac{1}{4}\exp(c+d)\left(1-\exp(-2c)-\exp(-2c-2d)\right)\\ &\geq \frac{1}{4}\exp(c+d)\left(1-\exp(-2c_0)-\exp(-4c_0)\right).\end{aligned}$$

By taking the logarithm of the two inequalities we therefore get

$$\ln(\frac{1}{4})(a+b)\ln(1+\exp(-2c_0)) \geq \ln(\frac{1}{4})(c+d)\ln(1-\exp(-2c_0)-\exp(-4c_0))$$

and

$$C = \ln\left(\frac{1+\exp(-2c_0)}{1-\exp(-2c_0)-\exp(-4c_0)}\right) > 0.$$

Note that

$$\lim_{c_0\to\infty}\ln\left(\frac{1+\exp(-2c_0)}{1-\exp(-2c_0)-\exp(-4c_0)}\right) = 0.$$

□

Definition 34. *Let $K \geq 1$. A triple $(b_1, b_2, b_3) \in \mathbb{R}^3$ of positive reals satisfies the K*-quasi-metric inequality *if the two largest members of the triple coincide up to a multiplicative error $\leq K$. i.e. if*

$$\frac{1}{K} \leq \frac{a}{b} \leq K$$

for the largest two elements a, b in (b_1, b_2, b_3). We denote this by $a \asymp b$ or $a \asymp_K b$.

Lemma 8 (This is based on Lemma 5.5 from [LS07], additional cases had to be considered for the proof to be complete)**.** *Let (X, d) be a δ-hyperbolic space, with $0 \leq \delta < \ln(2)$. There exists $s_0 > 0$ with the following property: Let $x, y, z \in X$ such that*

$$d(x,o), d(y,o), d(z,o), d(x,y), d(y,z), d(x,z) \geq s_0,$$

then $(\rho(x,y), \rho(y,z), \rho(z,x))$ satisfies the 2-quasi-metric inequality.

Proof. Without loss of generality assume that $\rho(x,y) \geq \rho(y,z) \geq \rho(z,x)$. This is equivalent to

$$\begin{aligned}\sinh\left(\frac{d(x,y)}{2}\right)\cosh\left(\frac{d(o,z)}{2}\right) &\geq \sinh\left(\frac{d(y,z)}{2}\right)\cosh\left(\frac{d(o,x)}{2}\right)\\ &\geq \sinh\left(\frac{d(z,x)}{2}\right)\cosh\left(\frac{d(o,y)}{2}\right).\end{aligned}$$

We know that for any $\zeta > 1$ there exists $s_0 > 0$ such that for all $t \geq s_0$ we have

$$\frac{1}{\zeta}\frac{1}{2}\exp(t) \leq \sinh(t) \leq \zeta\frac{1}{2}\exp(t)$$

with $\zeta = \frac{1}{1-\exp(-2s_0)}$ because of $\sinh(t) = \frac{1}{2}(\exp(t) - \exp(-t)) = \frac{1}{2}\exp(t)(1-\exp(-2t))$. And a similar inequality holds for $\cosh(x)$ with $\zeta' = (1+\exp(-2t))$:

$$\frac{1}{\zeta'}\frac{1}{2}\exp(x) \leq \cosh(t) \leq \zeta'\frac{1}{2}\exp(t)$$

because of

$$\cosh(t) = \frac{1}{2}(\exp(t) + \exp(-t)) = \frac{1}{2}\exp(t)(1+\exp(-2t)).$$

We can combine the above to get

$$\zeta^{-2}\zeta'^{-2}\frac{\exp\left(\frac{d(x,y)}{2}\right)\exp\left(\frac{d(o,z)}{2}\right)}{\exp\left(\frac{d(y,z)}{2}\right)\exp\left(\frac{d(o,x)}{2}\right)} \leq \frac{\sinh\left(\frac{d(x,y)}{2}\right)\cosh\left(\frac{d(o,z)}{2}\right)}{\sinh\left(\frac{d(y,z)}{2}\right)\cosh\left(\frac{d(o,x)}{2}\right)}$$
$$\leq \zeta^{2}\zeta'^{2}\frac{\exp\left(\frac{d(x,y)}{2}\right)\exp\left(\frac{d(o,z)}{2}\right)}{\exp\left(\frac{d(y,z)}{2}\right)\exp\left(\frac{d(o,x)}{2}\right)}$$

and therefore

$$\zeta^{-2}\zeta'^{-2}\exp\left(\frac{d(x,y)+d(o,z)-d(y,z)-d(o,x)}{2}\right)$$
$$\leq \frac{\sinh\left(\frac{d(x,y)}{2}\right)\cosh\left(\frac{d(o,z)}{2}\right)}{\sinh\left(\frac{d(y,z)}{2}\right)\cosh\left(\frac{d(o,x)}{2}\right)}$$
$$\leq \zeta^{2}\zeta'^{2}\exp\left(\frac{d(x,y)+d(o,z)-d(y,z)-d(o,x)}{2}\right).$$

We now need to consider the following possible cases:

1. If $d(x,y)+d(o,z) \geq d(y,z)+d(o,x) \geq d(x,z)+d(o,y)$ then by the δ-hyperbolicity of X, the largest two entries differ by at most 2δ. Therefore

$$\zeta^{2}\zeta'^{2}\exp\left(\frac{d(x,y)+d(o,z)-d(y,z)-d(o,x)}{2}\right) \leq \zeta^{2}\zeta'^{2}\exp(\delta).$$

2. If $d(y,z)+d(o,x) \geq d(x,y)+d(o,z) \geq d(x,z)+d(o,y)$ or $d(y,z)+d(o,x) \geq d(x,z)+d(o,y) \geq d(x,y)+d(o,z)$ then

$$\zeta^{2}\zeta'^{2}\exp\left(\frac{d(x,y)+d(o,z)-d(y,z)-d(o,x)}{2}\right) \leq \zeta^{2}\zeta'^{2}.$$

3. If $d(x,y)+d(o,z) \geq d(x,z)+d(o,y) \geq d(y,z)+d(o,x)$ then by the previous lemma $d(y,z)+d(o,x)+C \geq d(x,z)+d(o,y)$ and therefore

$$d(x,y)+d(o,z)-d(y,z)-d(o,x) \leq 2\delta + C.$$

4. If $d(x,z) + d(o,y) \geq d(x,y) + d(o,z) \geq d(y,z) + d(o,x)$ then $d(x,y) + d(o,z) - d(y,z) - d(o,x) \leq d(x,z) + d(o,y) - d(y,z) - d(o,x)$ and by the previous lemma:

$$d(x,z) + d(o,y) - d(y,z) - d(o,x) \leq C.$$

5. If $d(x,z) + d(o,y) \geq d(y,z) + d(o,x) \geq d(x,y) + d(o,z)$ then

$$d(x,y) + d(o,z) - d(y,z) - d(o,x) \leq 0.$$

In summary we get

$$\zeta^2\zeta'^2 \exp\left(\frac{d(x,y) + d(o,z) - d(y,z) - d(o,x)}{2}\right) \leq \zeta^2\zeta'^2 \exp\left(\frac{2\delta + C}{2}\right) = \zeta^2\zeta'^2 \exp\left(\delta + \frac{C}{2}\right).$$

Because we can choose s_0 such that $C > 0$ becomes arbitrarily close to 0 and $\zeta, \zeta' > 1$ arbitrarily close to 1, this completes the proof. □

Remark 17. *We show in the appendix (Corollary 4) that* $\ln(2)$ *is optimal.*

The following lemma is due to Frink [Fri37]:

Lemma 9. *Let* (X, ρ) *be a space with a distance function satisfying the following conditions for all* $x, y, z \in X$*:*

1. $\rho(x,y) = 0$ *iff* $x = y$,
2. $\rho(x,y) = \rho(y,x)$,
3. *For any* $\epsilon > 0$, $\rho(x,y) < \epsilon$ *and* $\rho(y,z) < \epsilon$ *together imply* $\rho(x,z) < 2\epsilon$.

Then the following holds for any chain $x = x_0, \dots, x_n = y$ *in X:*

$$\frac{1}{4}\rho(x,y) \leq \frac{1}{2}\rho(x,x_1) + \rho(x_1,x_2) + \dots + \rho(x_{n-2},x_{n-1}) + \frac{1}{2}\rho(x_{n-1},y) \leq \sum_{i=0}^{n-1} \rho(x_i,x_{i+1}).$$

Remark 18. *In particular this lemma applies to a distance function satisfying the* 2*-quasi-metric-inequality.*

Remark 19. *Note that the lemma does no longer work if we only have* $\rho(x,z) < \lambda\epsilon$ *for* $\lambda > 2$ *in the third condition above. To see this consider the following example: Consider the graph constructed as follows. Start with two vertices* v_0, v_1 *and set* $d(v_0, v_1) = x_0$*. Then connect both vertices to a third vertex* v_2 *to form a triangle such that the new sides have distance* $d(v_0, v_2) = d(v_1, v_2) = x_1$*, continue in the same fashion extending each side with vertices to form triangles. If we let*

$$x_0 = 1, x_1 = \frac{1}{(2+\mu)}, x_2 = \frac{1}{(2+\mu)^2}, \dots, x_i = \frac{1}{(2+\mu)^i}, \dots$$

then the condition would be satisfied for $\lambda = 2 + \mu$ *and* $\mu > 0$*. However applying the chain construction we get:*

$$\inf \sum d(v_i, v_{i+1}) \leq \lim_{n\to\infty} 2^n \frac{1}{(2+\mu)^n} = 0.$$

Proposition 22. *Let* $X = (X, d)$ *be a metric space. Fix* $o \in X$*. Let* $d_{rad}(x,y) = |d(x,o) - d(y,o)|$*. Then* (X, d_{rad}) *is a pseudo-metric space.*

Proof. Let $x, y, z \in X$ then we calculate:

$$d_{\text{rad}}(x,y) + d_{\text{rad}}(y,z) = |d(x,o) - d(y,o)| + |d(y,o) - d(z,o)|.$$

The following cases are possible:

1. $|d(x,o) - d(y,o)| + |d(y,o) - d(z,o)| = d(x,o) - d(y,o) + d(y,o) - d(z,o)$: This case is trivial.

2. $|d(x,o) - d(y,o)| + |d(y,o) - d(z,o)| = d(x,o) - d(y,o) + d(z,o) - d(y,o)$: Here we know that we must have $d(y,o) \leq d(x,o)$. Therefore $|d(z,o) - d(y,o)| \geq |d(z,o) - d(x,o)|$.

3. $|d(x,o) - d(y,o)| + |d(y,o) - d(z,o)| = d(y,o) - d(x,o) + d(y,o) - d(z,o)$: We know that we must have $d(y,o) \geq d(x,o)$ and $d(y,o) \geq d(z,o)$ and therefore either $d(y,o) \geq d(z,o) \geq d(x,o)$ or $d(y,o) \geq d(x,o) \geq d(z,o)$. But then the result follows from either $|d(y,o) - d(x,o)| \geq |d(z,o) - d(x,o)|$ or $|d(y,o) - d(z,o)| \geq |d(x,o) - d(z,o)|$.

4. $|d(x,o) - d(y,o)| + |d(y,o) - d(z,o)| = d(y,o) - d(x,o) + d(z,o) - d(y,o)$: This case is trivial.

The remaining cases are

$$\begin{aligned} d_{\text{rad}}(x,o) + d_{\text{rad}}(o,y) &= d(x,o) + d(o,y) \\ &\geq d(x,y) \\ &\geq |d(x,o) - d(y,o)| \\ &= d_{\text{rad}}(x,y), \end{aligned}$$

and

$$\begin{aligned} d_{\text{rad}}(o,x) + d_{\text{rad}}(x,y) &= d(o,x) + |d(x,o) - d(y,o)| \\ &\geq d(y,o) \\ &= d_{\text{rad}}(o,y). \end{aligned}$$

Therefore (X, d_{rad}) is a pseudo-metric space. □

Lemma 10. *Let $X = (X,d)$ be a metric space. Fix $o \in X$ and $\kappa < 0$. Let $d_{rad}(x,y) = |d(x,o) - d(y,o)|$ be the radial pseudo-metric. Then*

$$\rho_{rad}(x,y) := F_\kappa(d_{rad}(x,y); d_{rad}(x,o), d_{rad}(y,o))$$

is a pseudo-metric.

Proof. We have to show that:

$$\frac{\operatorname{sn}_\kappa\left(\frac{|d(x,o)-d(z,o)|}{2}\right)}{\operatorname{cs}_\kappa\left(\frac{d(x,o)}{2}\right)\operatorname{cs}_\kappa\left(\frac{d(z,o)}{2}\right)} \leq \frac{\operatorname{sn}_\kappa\left(\frac{|d(x,o)-d(y,o)|}{2}\right)}{\operatorname{cs}_\kappa\left(\frac{d(x,o)}{2}\right)\operatorname{cs}_\kappa\left(\frac{d(y,o)}{2}\right)} + \frac{\operatorname{sn}_\kappa\left(\frac{|d(y,o)-d(z,o)|}{2}\right)}{\operatorname{cs}_\kappa\left(\frac{d(y,o)}{2}\right)\operatorname{cs}_\kappa\left(\frac{d(z,o)}{2}\right)}.$$

By multiplying by $\operatorname{cs}_\kappa(d(x,o))\operatorname{cs}_\kappa(d(y,o))\operatorname{cs}_\kappa(d(z,o))$, we can rewrite this as follows:

$$\begin{aligned} &\operatorname{sn}_\kappa\left(\frac{|d(x,o) - d(z,o)|}{2}\right)\operatorname{cs}_\kappa\left(\frac{d(y,o)}{2}\right) \\ &\leq \operatorname{sn}_\kappa\left(\frac{|d(x,o) - d(y,o)|}{2}\right)\operatorname{cs}_\kappa\left(\frac{d(z,o)}{2}\right) + \operatorname{sn}_\kappa\left(\frac{|d(y,o) - d(z,o)|}{2}\right)\operatorname{cs}_\kappa\left(\frac{d(x,o)}{2}\right). \end{aligned}$$

We can further rewrite this by using the property that for any two $\alpha, \beta \in \mathbb{R}$ it holds that $\mathrm{sn}_k(\alpha)\,\mathrm{cs}_\kappa(\beta) = \frac{1}{2}(\mathrm{sn}_\kappa(\alpha+\beta) + \mathrm{sn}_\kappa(\alpha-\beta))$. This gives us the following inequality:

$$\begin{aligned}
&\mathrm{sn}_\kappa\left(\frac{|d(x,o)-d(z,o)|+d(y,o)}{2}\right)+\mathrm{sn}_\kappa\left(\frac{|d(x,o)-d(z,o)|-d(y,o)}{2}\right)\\
&\leq \mathrm{sn}_\kappa\left(\frac{|d(x,o)-d(y,o)|+d(z,o)}{2}\right)+\mathrm{sn}_\kappa\left(\frac{|d(x,o)-d(y,o)|-d(z,o)}{2}\right)\\
&+\mathrm{sn}_\kappa\left(\frac{|d(y,o)-d(z,o)|+d(x,o)}{2}\right)+\mathrm{sn}_\kappa\left(\frac{|d(y,o)-d(z,o)|-d(x,o)}{2}\right).
\end{aligned} \tag{4.15}$$

We now consider the following cases:

1. If $d(x,o) \geq d(y,o)$ and $d(z,o) \geq d(y,o)$ we can assume without loss of generality that $d(x,o) \geq d(z,o)$. In particular we get

$$\frac{\mathrm{sn}_\kappa\left(\frac{|d(x,o)-d(z,o)|}{2}\right)}{\mathrm{cs}_\kappa\left(\frac{d(x,o)}{2}\right)\mathrm{cs}_\kappa\left(\frac{d(z,o)}{2}\right)} \quad \leq \quad \frac{\mathrm{sn}_\kappa\left(\frac{|d(x,o)-d(y,o)|}{2}\right)}{\mathrm{cs}_\kappa\left(\frac{d(x,o)}{2}\right)\mathrm{cs}_\kappa\left(\frac{d(y,o)}{2}\right)},$$

   and we are done.

2. If $d(x,o) \geq d(y,o) \geq d(z,o)$, we can simplify Equation 4.15 as follows:

$$\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(z,o)+d(y,o)}{2}\right)+\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(z,o)-d(y,o)}{2}\right)$$
$$\leq \operatorname{sn}_\kappa\left(\frac{d(x,o)-d(y,o)+d(z,o)}{2}\right)+\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(y,o)-d(z,o)}{2}\right)$$
$$+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(z,o)+d(x,o)}{2}\right)+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(z,o)-d(x,o)}{2}\right)$$
$$\Longleftrightarrow$$
$$\cancel{\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(z,o)+d(y,o)}{2}\right)}+\cancel{\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(z,o)-d(y,o)}{2}\right)}$$
$$\leq \operatorname{sn}_\kappa\left(\frac{d(x,o)-d(y,o)+d(z,o)}{2}\right)+\cancel{\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(y,o)-d(z,o)}{2}\right)}$$
$$+\cancel{\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(z,o)+d(x,o)}{2}\right)}+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(z,o)-d(x,o)}{2}\right)$$
$$\Longleftrightarrow$$
$$0 \leq \operatorname{sn}_\kappa\left(\frac{d(x,o)-d(y,o)+d(z,o)}{2}\right)+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(z,o)-d(x,o)}{2}\right).$$

This is true because

$$\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(y,o)+d(z,o)}{2}\right) = -\operatorname{sn}_\kappa\left(\frac{-(d(x,o)-d(y,o)+d(z,o))}{2}\right) = -\operatorname{sn}_\kappa\left(\frac{-d(x,o)+d(y,o)-d(z,o)}{2}\right).$$

3. It remains to show the case $d(y,o) \geq d(x,o) \geq (z,o)$. In this case we can again rewrite Equation 4.15 as follows:

$$\begin{aligned}
&\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(z,o)+d(y,o)}{2}\right)+\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(z,o)-d(y,o)}{2}\right)\\
&\leq \operatorname{sn}_\kappa\left(\frac{d(y,o)-d(x,o)+d(z,o)}{2}\right)+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(x,o)-d(z,o)}{2}\right)\\
&+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(z,o)+d(x,o)}{2}\right)+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(z,o)-d(x,o)}{2}\right)\\
&\quad\Longleftrightarrow\\
&\operatorname{sn}_\kappa\left(\frac{d(x,o)-d(z,o)-d(y,o)}{2}\right)\\
&\quad\leq \operatorname{sn}_\kappa\left(\frac{d(y,o)-d(x,o)+d(z,o)}{2}\right)\\
&\qquad+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(x,o)-d(z,o)}{2}\right)\\
&\qquad+\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(z,o)-d(x,o)}{2}\right).
\end{aligned}$$

This further simplifies to

$$0 \leq 2\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(x,o)+d(z,o)}{2}\right)+2\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(x,o)-d(z,o)}{2}\right).$$

If $d(y,o)-d(x,o)-d(z,o) \geq 0$ we are done. Otherwise we have to show that

$$0 \leq 2\operatorname{sn}_\kappa\left(\frac{d(y,o)-d(x,o)+d(z,o)}{2}\right)-2\operatorname{sn}_\kappa\left(\frac{d(x,o)+d(z,o)-d(y,o)}{2}\right).$$

This is true whenever $d(y,o)-d(x,o)+d(z,o) \geq d(x,o)+d(z,o)-d(y,o)$. Note that $d(x,o)-d(y,o) \leq d(y,o)-d(x,o)$ because of the assumption of this case. Therefore we are done.

Note that in the other cases where some terms are zero, ρ_{rad} also satisfies the triangle inequality: Whenever $d_{\text{rad}}(x,y)=0$ it follows that $d(x,o)=d(y,o)$. In particular we get:

$$\rho_{\text{rad}}(x,z) \leq \rho_{\text{rad}}(x,y)+\rho_{\text{rad}}(y,z)=0+\rho_{\text{rad}}(x,z).$$

□

Lemma 11. *Let $X = (X, d)$ be a metric space and $o \in X$. Let $\kappa < 0$ and*

$$\rho(x, y) := F_\kappa(d(x, y); d(x, o), d(y, o)).$$

Furthermore let

$$\rho_{rad}(x, y) := F_\kappa(|d(x, o) - d(y, o)|; d(x, o), d(y, o)).$$

Let $x = x_0, \ldots, x_n = y$ be a chain between two points $x, y \in X$. Let $R = \max_i\{d(o, x_i)\}$ and let $k \in \{1, \ldots, n\}$. Then the following holds:

1. $$\rho_{rad}(x, y) \leq \rho(x, y),$$

2. $$\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) \geq \rho_{rad}(x, x_k) + \rho_{rad}(x_k, y),$$

3. $$\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) \geq \frac{d(x, y)/2}{\mathrm{cs}_\kappa(R/2)^2}.$$

Proof. The first inequality follows directly from $|d(x, o) - d(y, o)| \leq d(x, y)$. For the second one note that by Lemma 10,

$$\rho_{\mathrm{rad}}(x, x_k) + \rho_{\mathrm{rad}}(x_k, y) \leq \sum_{i=0}^{n-1} \rho_{\mathrm{rad}}(x_i, x_{i+1}).$$

The last inequality follows from the following calculation:

$$\begin{aligned}\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) &\geq \sum_{i=0}^{n-1} \frac{\mathrm{sn}_\kappa(d(x_i, x_{i+1})/2)}{\mathrm{cs}_\kappa(R/2)^2} \\ &\geq \sum_{i=0}^{n-1} \frac{d(x_i, x_{i+1})/2}{\mathrm{cs}_\kappa(R/2)^2} \\ &\geq \frac{d(x, y)/2}{\mathrm{cs}_\kappa(R/2)^2}.\end{aligned}$$

Here we used that $t \leq \sinh(t)$ for $t \geq 0$. □

Remark 20. *If* (X, d) *is a bounded metric space with* $\operatorname{diam}(X) \leq \frac{\pi}{\sqrt{\kappa}}$ *for some* $\kappa > 0$*, then the following holds:*

1. $$\rho_{rad}(x, y) \leq \rho(x, y),$$

2. $$\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) \geq \rho_{rad}(x, x_k) + \rho_{rad}(x_k, y),$$

3. $$\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) \geq \frac{d(x, y)/2}{C \operatorname{cs}_\kappa(R/2)^2}.$$

Where $\infty > C > 0$ *is some constant only depending on* κ*. This comes from requiring* $\frac{t}{C} \leq \sin(t)$ *for all* $t \in [0, \pi/2]$.

Lemma 12. *We have the following relations:*

1. *For* $0 \leq a, b$ *and* $\kappa < 0$ *we have*
$$\operatorname{cs}_\kappa(a + b) \leq 2 \operatorname{cs}_\kappa(a) \operatorname{cs}_\kappa(b).$$

2. *For* $0 \leq a, b \leq \frac{\pi}{2\sqrt{-\kappa}}$ *and* $\kappa > 0$ *we have:*
$$\operatorname{cs}_\kappa(a + b) \leq 2 \operatorname{cs}_\kappa(a) \operatorname{cs}_\kappa(b).$$

3. *For* $0 \leq a, b$*,* $0 < s \leq a$ *and* $\kappa < 0$ *we have:*
$$\operatorname{sn}_\kappa(a + s) \leq 2 \operatorname{cs}_\kappa(s) \operatorname{sn}_\kappa(a).$$

4. *For* $0 \leq a, b$ *and* $\kappa < 0$ *we have:* $\operatorname{cs}_\kappa(a - s) \geq \frac{1}{2 \operatorname{cs}_\kappa(s)} \operatorname{cs}_\kappa(a)$.

5. *For* $0 \leq a, b$*,* $0 < 2s \leq a$ *and* $\kappa < 0$ *we have:*
$$\operatorname{sn}_\kappa(a - s) \geq \frac{1}{2 \operatorname{cs}_\kappa(s)} \operatorname{sn}_\kappa(a).$$

Proof. Writing out the definitions we get:

$$\begin{aligned}
2\,\mathrm{cs}_\kappa(a)\,\mathrm{cs}_\kappa(b) &= 2\cosh(\sqrt{-\kappa}a)\cosh(\sqrt{-\kappa}b)\\
&= 2\cdot\frac{e^{\sqrt{-\kappa}a}+e^{-\sqrt{-\kappa}a}}{2}\cdot\frac{e^{\sqrt{-\kappa}b}+e^{-\sqrt{-\kappa}b}}{2}\\
&= 2\cdot\frac{e^{\sqrt{-\kappa}(a+b)}+e^{\sqrt{-\kappa}(a-b)}+e^{\sqrt{-\kappa}(b-a)}+e^{\sqrt{-\kappa}(-a-b)}}{4}\\
&\geq \frac{e^{\sqrt{-\kappa}(a+b)}+e^{\sqrt{-\kappa}(-a-b)}}{2}\\
&= \mathrm{cs}_\kappa(a+b).
\end{aligned}$$

The other equations follow in a similar way. □

In the following let

$$\bar\rho(x,y) := \inf\{\sum_{i=0}^{n}\rho(x_i,x_{i+1})\,|\,x=x_0,\dots,x_n=y\},$$

where the infimum is taken over all finite chains between x and y.

Lemma 13 (Lemma 5.3 in [LS07]). *Let (X,d) be a general metric space. For all $s>0$ there exists $\mu_1(s)>0$ such that for all $x,y\in X$ with $d(x,y)\leq s$ we have*

$$\bar\rho(x,y)\geq\mu_1(s)\rho(x,y).$$

In particular for any $x\neq y$, $\bar\rho(x,y)>0$.

Proof. Let $x,y\in X$ and let $r_x=d(o,x)$ and $r_y=d(o,y)$. Without loss of generality assume that $r_x\leq r_y$. Let $x=x_0,\dots,x_n=y$ be a chain from x to y and let $R=\max_i\{d(o,x_i)\}$ and k such that $d(o,x_k)=R$.

Let $F_\kappa^{\mathrm{rad}}(s,t)=F_\kappa(|s-t|,s,t)$. Two cases are possible:

1. $R \geq r_y + s$: In this case we can apply the second inequality from Lemma 11 to get

$$\begin{aligned}
\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) &\geq \rho_{\mathrm{rad}}(x, x_k) + \rho_{\mathrm{rad}}(x_k, y) \\
&= F_\kappa^{\mathrm{rad}}(R, r_x) + F_\kappa^{\mathrm{rad}}(R, r_y) \\
&\geq F_\kappa^{\mathrm{rad}}(r_x + s, r_x) + F_\kappa^{\mathrm{rad}}(r_y + s, r_y) \\
&= \frac{\mathrm{sn}_\kappa(s/2)}{\mathrm{cs}_\kappa(r_x/2)\,\mathrm{cs}_\kappa((r_x+s)/2)} + \frac{\mathrm{sn}_\kappa(s/2)}{\mathrm{cs}_\kappa(r_y/2)\,\mathrm{cs}_\kappa((r_y+s)/2)} \\
&\geq \frac{\mathrm{sn}_\kappa(s/2)}{2\,\mathrm{cs}_\kappa(r_x/2)\,\mathrm{cs}_\kappa(r_y/2)\,\mathrm{cs}_\kappa(s/2)} + \frac{\mathrm{sn}_\kappa(s/2)}{2\,\mathrm{cs}_\kappa(r_x/2)\,\mathrm{cs}_\kappa(r_y/2)\,\mathrm{cs}_\kappa(s)} \\
&= \left(\frac{1}{2\,\mathrm{cs}_\kappa(s/2)} + \frac{1}{2\,\mathrm{cs}_\kappa(s)}\right)\rho(x, y) \\
&\geq \frac{1}{\mathrm{cs}_\kappa(s)}\rho(x, y).
\end{aligned}$$

   Note that from $r_x + s \geq r_x + d(x, y) \geq r_y$ we get $(r_y + s)/2 \leq r_x/2 + s$ and by applying Lemma 12 we get

$$\mathrm{cs}_\kappa((r_y + s)/2) \leq 2\,\mathrm{cs}_\kappa(r_x/2)\,\mathrm{cs}_\kappa(s).$$

   This was applied in the second last inequality above.

   Note that this case still holds for $\kappa > 0$ if $\mathrm{diam}(X) \leq \frac{\pi}{2\sqrt{\kappa}}$.

2. $R \leq r_y + s$: In the case that $R \leq r_y + s$ we have $r_x \leq r_y$ and from the triangle inequality we get $R \leq r_y + s \leq r_x + 2s$. Applying Lemma 12 we get $\mathrm{cs}_\kappa(R/2) \leq 2\,\mathrm{cs}_\kappa(s)\,\mathrm{cs}_\kappa(r_x/2)$ and $\mathrm{cs}_\kappa(R/2) \leq 2\,\mathrm{cs}_\kappa(s/2)\,\mathrm{cs}_\kappa(r_y/2)$. We apply (3) from Lemma 11 and get:

$$\begin{aligned}
\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) &\geq \frac{d(x, y)/2}{\mathrm{cs}_\kappa(R/2)^2} \\
&\geq \frac{d(x, y)/2}{4\,\mathrm{cs}_\kappa(s)\,\mathrm{cs}_\kappa(r_x/2)\,\mathrm{cs}_\kappa(s/2)\,\mathrm{cs}_\kappa(r_y/2)} \\
&\geq \frac{d(x, y)/2}{4\,\mathrm{cs}_\kappa(s)\,\mathrm{cs}_\kappa(s/2)\,\mathrm{sn}_\kappa(d(x, y)/2)}\rho(x, y).
\end{aligned}$$

In particular we get:

$$\mu_1(d(x,y)) = \begin{cases} \dfrac{1}{\operatorname{cs}_\kappa(d(x,y))}, & \text{if } R \geq r_y + d(x,y), \\ \dfrac{d(x,y)/2}{4\operatorname{cs}_\kappa(d(x,y))\operatorname{cs}_\kappa(d(x,y)/2)\operatorname{sn}_\kappa(d(x,y)/2)} & \text{if } R < r_y + d(x,y). \end{cases}$$

In case that $\kappa > 0$ and $\operatorname{diam}(X) \leq \frac{\pi}{2\sqrt{\kappa}}$ we still get:

$$\mu_1(d(x,y)) = \begin{cases} \dfrac{1}{\operatorname{cs}_\kappa(d(x,y))}, & \text{if } R \geq r_y + d(x,y), \\ \dfrac{d(x,y)/2}{4C\operatorname{cs}_\kappa(d(x,y))\operatorname{cs}_\kappa(d(x,y)/2)\operatorname{sn}_\kappa(d(x,y)/2)} & \text{if } R < r_y + d(x,y), \end{cases}$$

where $C > 0$ is some finite positive constant depending only on κ.

We can therefore take (for $\kappa < 0$)

$$\mu_1(s) = \min\left\{\frac{1}{\operatorname{cs}_\kappa(s)}, \frac{s/2}{4\operatorname{cs}_\kappa(s)\operatorname{cs}_\kappa(s/2)\operatorname{sn}_\kappa(s/2)}\right\} = \frac{s/2}{4\operatorname{cs}_\kappa(s)\operatorname{cs}_\kappa(s/2)\operatorname{sn}_\kappa(s/2)}.$$

Note that $\mu_1(s)$ is decreasing in s therefore it follows that for $d(x,y) \leq s$ we have

$$\bar{\rho}(x,y) \geq \mu_1(s)\rho(x,y).$$

□

The following theorem then follows directly from the previous lemmas:

Theorem 17 (Theorem 5.2 (A) in [LS97]). *Let (X,d) be a metric space and $o \in X$. Let $\kappa < 0$ and*

$$\rho(x,y) := F_\kappa(d(x,y); d(x,o), d(y,o)).$$

Then ρ is symmetric, positive and metrizable. That is

$$\bar{\rho}(x,y) = \inf \sum_{i=0}^{n-1} \rho(x_i, x_{i+1}),$$

where the infimum is taken over all sequences $x = x_0, \ldots, x_n = y$ (for all $n < \infty$), is a metric. □

Lemma 14. *Let $\kappa > 0$ and let (X, d) be a metric space with* $\operatorname{diam}(X) \leq \frac{\pi}{\sqrt{\kappa}}$. *Set*

$$\rho(x,y) := F_\kappa(d(x,y); d(x,o), d(y,o)).$$

Then ρ is symmetric, positive and bi-Lipschitz-metrizable. That is

$$\bar{\rho}(x,y) = \inf \sum_{i=0}^{n-1} \rho(x_i, x_{i+1}),$$

where the infimum is taken over all sequences $x = x_0, \ldots, x_n = y$ (for all $n < \infty$), is a metric. Furthermore this new metric is bi-Lipschitz to ρ:

$$\frac{\sqrt{\kappa}}{\pi}\rho \leq \bar{\rho} \leq \rho.$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) &= \sum_{i=0}^{n-1} \frac{\operatorname{sn}_\kappa(d(x_i, x_{i+1})/2)}{\operatorname{cs}_\kappa(d(x_i, o)/2)\operatorname{cs}_\kappa(d(x_{i+1}, o)/2)} \\
&= \sum_{i=0}^{n-1} \frac{\sin(\sqrt{\kappa} d(x_i, x_{i+1})/2)}{\sqrt{\kappa}\cos(\sqrt{\kappa} d(x_i, o)/2)\cos(\sqrt{\kappa} d(x_{i+1}, o)/2)} \\
&\geq \frac{1}{\sqrt{\kappa}} \sum_{i=0}^{n-1} \sin(\sqrt{\kappa} d(x_i, x_{i+1})/2) \\
&\geq \frac{\sqrt{\kappa}}{\pi} \sum_{i=0}^{n-1} d(x_i, x_{i+1})/2 \\
&\geq \frac{\sqrt{\kappa}}{\pi} d(x,y).
\end{aligned}$$

□

Lemma 15 (Lemma 5.6 in [LS07])**.** *Let $\kappa < 0$. For all $s > 0$ there exists $\mu_2(s) > 0$, such that the following holds in an arbitrary metric space X. Let $x, y, z \in X$ with $d(x,y) \leq s$ and $d(x,z) > s$. Then*

$$\mu_2(s)\rho(x,z) \leq \rho(x,y) + \rho(y,z).$$

Proof. Consider the two cases:

1. $d(x,z) \leq 2s$. In this case we can use Lemma 13 and take $\mu_2 = \mu_1(2s)$. Then we have

$$\mu_1(2s)\rho(x,z) \leq \bar{\rho}(x,z) \leq \rho(x,y) + \rho(y,z).$$

2. $d(x,z) > 2s$: Let $r_x = d(o,x), r_y = d(o,y), r_z = d(o,z)$. We have the following relations:

$$s + d(y,z) \geq d(x,y) + d(y,z) \geq d(x,z) > 2s.$$

Therefore we can apply Lemma 12 to get:

$$\operatorname{sn}_\kappa(d(y,z)/2) \geq \frac{\operatorname{sn}_\kappa(d(y,z)/2 + s/2)}{2\operatorname{cs}_\kappa(s/2)}$$

and

$$\operatorname{cs}_\kappa(r_z/2) \leq 2\operatorname{cs}_\kappa(r_z/2 - s/2)\operatorname{cs}_\kappa(s/2).$$

Using those inequalities we can approximate:

$$\begin{aligned}
\rho(y,z) &= \frac{\operatorname{sn}_\kappa(d(y,z)/2)}{\operatorname{cs}_\kappa(r_y/2)\operatorname{cs}_\kappa(r_z/2)} \\
&\geq \frac{1}{4\operatorname{cs}_\kappa(s/2)^2}\frac{\operatorname{sn}_\kappa(d(y,z)+s)/2)}{\operatorname{cs}_\kappa(r_y/2)\operatorname{cs}_\kappa((r_z-s)/2)} \\
&= \frac{1}{4\operatorname{cs}_\kappa(s/2)^2}\frac{\operatorname{sn}_\kappa(d(y,z)+d(x,y))/2)}{\operatorname{cs}_\kappa(r_y/2)\operatorname{cs}_\kappa((r_z-d(x,y))/2)} \\
&\geq \frac{1}{4\operatorname{cs}_\kappa(s/2)^2}\rho(x,z).
\end{aligned}$$

It follows that

$$\rho(x,y) + \rho(y,z) \geq \frac{1}{4\operatorname{cs}_\kappa(s/2)^2}\rho(x,z).$$

We can therefore set

$$\mu_2 := \min\left(\mu_1(2s), \frac{1}{4\operatorname{cs}_\kappa(s/2)^2}\right)$$

to satisfy the inequality.

□

We now can give the proof of Theorem 16.

Proof of Theorem 16 ([LS07]). Assume that (X, d) is δ-hyperbolic with $0 \leq \delta < \ln(2)$. Let $s_0 = s_0(\delta) > 0$ be the constant given in Lemma 8 and let $x, y \in X$ and $x = x_0, \dots, x_n = y$ be a chain from x to y. Let $s = d(x, y), r_x = d(o, x), r_y = d(o, y)$ and without loss of generality assume that $r_x \leq r_y$. We consider two cases

1. There exists some $k \in \{0, \dots, n\}$ with $d(o, x_k) \leq s_0$: If $r_y \leq 2s_0$ then we obtain $d(x, y) \leq r_x + r_y \leq 4s_0$ we can then apply Lemma 13 and get

$$\sum \rho(x_i, x_{i+1}) \geq \mu_1(4s_0)\rho(x, y).$$

In case that $r_y > 2s_0$ compute using Lemma 12:

$$\begin{aligned} \operatorname{sn}_\kappa(\frac{r_y - s_0}{2}) &\geq \frac{1}{2\operatorname{cs}_\kappa(s_0/2)} \operatorname{sn}_\kappa(\frac{r_y}{2}) \\ &= \frac{1}{2\operatorname{cs}_\kappa(s_0/2)} \operatorname{sn}_\kappa(\frac{r_y + r_x}{2} - \frac{r_x}{2}) \\ &\geq \frac{1}{4\operatorname{cs}_\kappa(s_0/2)\operatorname{cs}_\kappa(r_x/2)} \operatorname{sn}_\kappa(\frac{r_y + r_x}{2}). \end{aligned}$$

Now we use Lemma 11 ($\rho_{\text{rad}}(\cdot,\cdot) \leq \rho(\cdot,\cdot)$) and the triangle inequality for ρ_{rad} (Lemma 10) to approximate

$$\begin{aligned}
\sum_i \rho(x_i, x_{i+1}) &\geq \rho_{\text{rad}}(x, x_k) + \rho_{\text{rad}}(x_k, y) \\
&\geq \rho_{\text{rad}}(x_k, y) \\
&= F_\kappa(|r_y - r_{x_k}|; r_y, r_{x_k}) \\
&\geq F_\kappa(|r_y - s_0|; r_y, s_0) \\
&= \frac{\text{sn}_\kappa(\frac{r_y - s_0}{2})}{\text{cs}_\kappa(s_0/2)\,\text{cs}_\kappa(r_y/2)} \\
&\geq \frac{1}{4\,\text{cs}_\kappa(s_0/2)^2} \frac{\text{sn}_\kappa(\frac{r_y + r_x}{2})}{\text{cs}_\kappa(r_x/2)\,\text{cs}_\kappa(r_y/2)} \\
&\geq \frac{1}{4\,\text{cs}_\kappa(s_0/2)^2} \rho(x, y).
\end{aligned}$$

2. In case that $d(o, x_k) \geq s_0$ for all $k \in \{0, \dots, n\}$ do the following: Define a subsequence $0 \leq i_0 < i_1 < \dots < i_k = n$ recursively in the following way: Let $i_0 \in \{0, \dots, n\}$ be the largest number such that $d(x, x_{i_0}) \leq s_0$. If i_m is already defined and $i_m < n$, then let i_{m+1} be the largest number such that $d(x_{(i_m)+1}, x_{i_{(m+1)}}) \leq s_0$.

   For $j \in \{0, \dots, k\}$ define
   $$z_j := x_{i_j}$$
   and define
   $$w_j := x_{i_{(j-1)}+1}$$
   for $j \geq 1$ and set $w_0 := x$. By this construction the following inequalities hold for $i \neq j$:
   $$d(w_i, w_j) > s_0$$
   and
   $$d(w_j, z_j) \leq s_0.$$

Given a chain $x_0, \dots, x_n$ in X and two elements of the chain u, v use the following notation:

$$\sum_{u,\dots,v} \rho := \sum_{i=s}^{t-1} \rho(x_i, x_{i+1}),$$

where $0 \le s < t \le n$ are such that $x_s = u$ and $x_t = v$.

We have

$$\begin{aligned}
\sum_{x_0,\dots,x_n} \rho &= \sum_{i=0}^{k-1} \left(\sum_{w_i,\dots,w_{i+1}} \rho \right) + \sum_{w_k,\dots,z_k} \rho \\
&= \sum_{i=0}^{k-1} \left(\sum_{w_i,\dots,z_i} \rho + \rho(z_i, w_{i+1}) \right) + \sum_{w_k,\dots,z_k} \rho \\
&\ge \sum_{i=0}^{k-1} (\mu_1(s_0)\rho(w_i, z_i) + \rho(z_i, w_{i+1})) + \mu_1(s_0)\rho(w_k, z_k) \\
&\ge \sum_{i=0}^{k-1} (\mu_1(s_0)\mu_2(s_0)\rho(w_i, w_{i+1})) + \mu_1(s_0)\rho(w_k, y) \\
&\ge \frac{1}{4}\mu_1(s_0)\mu_2(s_0)\rho(x, w_k) + \mu_1(s_0)\rho(w_k, y) \\
&\ge \frac{1}{4}\mu_1(s_0)\mu_2(s_0)^2\rho(x, y).
\end{aligned}$$

Here we have applied Lemma 13 to get

$$\sum_{w_i,\dots,z_i} \rho \ge \bar{\rho}(w_i, z_i) \ge \mu_1(s_0)\rho(w_i, z_i)$$

and then used Lemma 15 to get

$$\begin{aligned}
\mu_1(s_0)\rho(w_i, z_i) + \rho(z_i, w_{i+1}) &\ge \mu_1(s_0)\left(\rho(w_i, z_i) + \rho(z_i, w_{i+1})\right) \\
&\ge \mu_1(s_0)\mu_2(s_0)\rho(w_i, w_{i+1}).
\end{aligned}$$

We have used that $\rho(w_i, z_i) \le s_0$ and $\rho(w_i, w_{i+1}) > s_0$. Note that we have $d(o, w_i), d(o, w_j), d(o, w_l), d(w_i, w_j), d(w_j, w_l), d(w_l, w_i) \ge s_0$ for all different $i, j, l \in \{0, \dots, k-1\}$. Therefore we can apply Lemma 8 and then apply Frink's Lemma 9. In the end we apply Lemma 15 again and the result follows.

□

Lemma 16. *For any complete δ-hyperbolic space (X, d) with base point $o \in X$ and $\bar{\rho}$ as defined previously. We can Cauchy complete the metric space as $(\bar{X} = X \cup \partial X, \bar{\rho})$ and we have $\partial_\infty X = \partial X$ as sets. Furthermore we have*

$$\omega \in \partial X \iff \bar{\rho}(o, \omega) = 1.$$

Proof. We proof a version of this result in the next section for general metric spaces. It follows that whenever the Gromov boundary $\partial_\infty X$ is well defined, then it is equal as a set to the set of Cauchy sequences. □

4.3.5 *General Metric Spaces*

In this section let (X, d) be a complete metric space, and $o \in X$ a base point. Define as in the previous section the semi-metric[6]

$$\rho(x,y) = \frac{\sinh(d(x,y)/2)}{\cosh(d(x,o)/2)\cosh(d(y,o)/2)},$$

and form the metric $\bar{\rho}$, by the chain construction

$$\bar{\rho}(x,y) = \inf \sum \rho(x_i, x_{i-1}),$$

as done previously. We can then form the metric completion (as in Proposition 2) $\bar{X} = X \cup \partial X$, by abuse of notation we write $\bar{\rho}$ for the metric on $\bar{X}$.

Proposition 23. *For a general metric space the topologies of* (X, d) *and* $(X, \bar{\rho})$ *are equivalent.*

Proof. 1. Let $x_0 \in X$, $\epsilon > 0$ and let $B = B_\epsilon(x_0, d)$ be the open ball around x_0 in the metric d. We have to show that there exists an η which depends only on x_0, ϵ such that the open ball $B' = B_\eta(x_0, \bar{\rho})$ in the metric $\bar{\rho}$ is contained in the ball B. Assume that $d(x_0, x) < \eta$ for some η to be determined later. Then it follows that

$$\begin{aligned}\bar{\rho}(x_0,x) \le \rho(x_0,x) &= \frac{\sinh(\frac{d(x_0,x)}{2})}{\cosh(\frac{d(x_0,o)}{2})\cosh(\frac{d(x,o)}{2})} \\ &\le \frac{\sinh(\frac{\eta}{2})}{\cosh(0)\cosh(0)} \\ &= \sinh(\frac{\eta}{2}).\end{aligned}$$

In particular we can choose $\eta < 2\,\mathrm{arcsinh}(\epsilon)$ and we get that $\bar{\rho}(x_0, x) < \epsilon$.

6 Again we drop the specification of the base point o from the metric whenever it is clear from the context.

2. To show the other direction let $x_0 \in X$, $\epsilon > 0$ and let $B = B_\epsilon(x_0, \bar{\rho})$ be the open ball around x_0 in the metric $\bar{\rho}$. Assume that $\bar{\rho}(x_0, x) < \eta$ for some η to be determined later. Remark: We first show that there exists a constant $c_1 > 0$ such that for all $x, y \in X$ with $d(x_0, x) < \epsilon$ and $d(x_0, y) > 2\epsilon$ it holds that $\rho(x, y) \geq c_1$. To see this note that

$$\begin{aligned}
\rho(x,y) &= \frac{\sinh(\frac{d(x,y)}{2})}{\cosh(\frac{d(x,o)}{2})\cosh(\frac{d(y,o)}{2})} \\
&\geq \frac{\sinh(\frac{d(x,y)}{2})}{\cosh(\frac{d(x,x_0)+d(x_0,o)}{2})\cosh(\frac{d(y,x)+d(x,o)}{2})} \\
&\geq \frac{\sinh(\frac{d(x,y)}{2})}{\cosh(\frac{d(x,x_0)+d(x_0,o)}{2})\cosh(\frac{d(x,y)+d(x,x_0)+d(x_0,o)}{2})} \\
&\geq \frac{\sinh(\frac{d(x,y)}{2})}{\cosh(\frac{\epsilon+d(x_0,o)}{2})\cosh(\frac{d(x,y)+\epsilon+d(x_0,o)}{2})} \\
&\geq c_1(\epsilon, x_0).
\end{aligned}$$

We consider the following cases:

a) Let $d(x_0, x) \leq 2\epsilon$. Then we have by Lemma 13 that

$$\begin{aligned}
\eta > \bar{\rho}(x_0,x) &\geq \mu_1(2\epsilon)\rho(x_0,x) \\
&= \mu_1(2\epsilon)\frac{\sinh(\frac{d(x_0,x)}{2})}{\cosh(\frac{d(x_0,o)}{2})\cosh(\frac{d(x,o)}{2})} \\
&\geq \mu_1(2\epsilon)\frac{\frac{d(x_0,x)}{2}}{\cosh(\frac{d(x_0,o)}{2})\cosh(\frac{d(x,x_0)+d(x_0,o)}{2})} \\
&\geq \mu_1(2\epsilon)\frac{\frac{d(x_0,x)}{2}}{\cosh(\frac{d(x_0,o)}{2})\cosh(\frac{2\epsilon+d(x_0,o)}{2})}.
\end{aligned}$$

In particular if we choose $\eta > 0$ such that

$$\epsilon > \frac{2\eta \cosh(\frac{d(x_0,o)}{2}) \cosh(\frac{2\epsilon + d(x_0,o)}{2})}{\mu_1(2\epsilon)}$$

we are done.

b) Let $d(x_0, x) > 2\epsilon$. Then there exists a chain $x_0, x_1, \dots, x_{n-1}, x_n = x$ such that $\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) < \eta$, because of $\bar{\rho}(x_0, x) < \eta$ such a chain clearly must exist. Let k be the largest index such that for all $i \in \{0, \dots, k\}$ we have $d(x_0, x_i) \leq 2\epsilon$, then: $\bar{\rho}(x_0, x_k) < \eta$ because $\bar{\rho}(x_0, x_k) \leq \sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) < \eta$. It then follows from the previous case that $d(x_0, x_k) < \epsilon$ for a suitable choice (as above) of η. By the remark above it follows further that $\rho(x_k, x_{k+1}) \geq c_1(\epsilon, x_0)$. But then $\eta \geq c_1$ by the chain construction. We can therefore take $0 < \eta < c_1$ and this case does not appear.

□

Proposition 24. *Let (X, d) be a metric space with base point $o \in X$ and let $\{x_i\} \subset X$ be a sequence of points converging to infinity (i.e., $\lim_{i,j\to\infty}(x_i|x_j)_o = \infty$). Then $\{x_i\}$ is a Cauchy sequence in $(X, \bar{\rho})$.*

Proof. We have

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty$$

therefore

$$\lim_{i,j\to\infty} \frac{1}{2}\Big(d(x_i, o) + d(x_j, o) - d(x_i, x_j)\Big) = \infty,$$

and so we also know that $d(x_i, o) \to \infty$ as $i \to \infty$. For $i, j \to \infty$ we then get that

$$\begin{aligned}
\bar{\rho}(x_i, x_j) \leq \rho(x_i, x_j) &= \frac{\sinh(d(x_i, x_j)/2)}{\cosh(d(x_i, o)/2)\cosh(d(x_j, o)/2)} \\
&\leq \frac{4\exp(d(x_i, x_j)/2)}{\exp(d(x_i, o)/2)\exp(d(x_j, o)/2)} \\
&= 4\exp(-(x_i|x_j)_o) \to 0.
\end{aligned}$$

□

Corollary 2. *Let (X, d) be a metric space with a base point $o \in X$ and let $\{x_i\}$ and $\{y_i\}$ be two sequences of points in X converging to infinity. If*

$$\lim_{i\to\infty}(x_i|y_i)_o = \infty$$

then both sequences converge to the same point in the space $(X, \bar{\rho})$.

Remark 21. *For a metric space (X, d) with base point $o \in X$ and a sequence of points $\{x_i\}$ in X converging to infinity we have*

$$\lim_{i\to\infty}\bar{\rho}(o, x_i) \leq 1.$$

Proof.

$$\begin{aligned}\bar{\rho}(o, x_i) \leq \rho(o, x_i) &= \frac{\sinh(d(o, x_i)/2)}{\cosh(d(o, o)/2)\cosh(d(x_i, o)/2)} \\ &= \frac{\sinh(d(o, x_i)/2)}{\cosh(d(x_i, o)/2)} \\ &= \tanh(d(x_i, o)/2) \to 1\end{aligned}$$

as $d(x_i, o) \to \infty$. □

Remark 22. *For a complete δ-hyperbolic space (X, d) with Gromov boundary $\partial_\infty X$, the above results imply that for some $x \in [x] \in \partial_\infty X$ (meaning for any choice of representative), $x \in \partial X$. Furthermore the corollary implies that for $x, x' \in [x] \in \partial_\infty X$, we have equality $x = x'$ in ∂X.*

Lemma 17. *Let (X, d) be a metric space with base point $o \in X$ and a point ω at infinity. Let $o = x_0, x_1, \ldots, x_n, x_{n+1} = \omega$ be a chain. Then*

$$\sum_{i=0}^{n}\rho(x_i, x_{i+1}) = \sum_{i=0}^{n}\frac{\sinh(d(x_i, x_{i+1}))}{\cosh(d(o, x_i)/2)\cosh(d(o, x_{i+1})/2)} \geq 1.$$

Proof. From Lemma 10 we know that the radial part of the metric ρ satisfies the triangle inequality. If we had $\bar{\rho}(o, \omega) < 1$ then there would be some chain $o = x_1, \ldots, x_n = \omega$ with

$$\sum_{i=1}^{n-1}\rho(x_i, x_{i+1}) < \rho(o, \omega) = 1.$$

But we can now calculate:

$$\sum_{i=1}^{n-1} \rho(x_i, x_{i+1}) \geq \sum_{i=1}^{n-1} \rho_{\text{rad}}(x_i, x_{i+1}) \geq \rho_{\text{rad}}(o, \omega) = \rho(o, \omega) = 1.$$

We therefore must have $\bar{\rho}(o, \omega) = 1$. □

In conclusion we have the following result:

Theorem 18. *Let (X, d) be a metric space with base point $o \in X$ and a point at infinity $\omega \in \partial X$. Then:*

$$\bar{\rho}(o, \omega) = 1.$$

In particular we also get the following theorem:

Theorem 19. *Let (X, d) be a metric space with base point $o \in X$ and let $x \in X \cup \partial X$ be some point. Then for the metric $\bar{\rho}$ with respect to the base point o the following holds:*

$$\bar{\rho}(o, x) = 1 \iff x \in \partial X$$

Proof. One direction directly follows from the above theorem. For the other direction ($x \in X \setminus \partial X$ and $d(o, x) < \infty$) note that:

$$\bar{\rho}(o, x) \leq \rho(o, x) = \rho_{\text{rad}}(o, x) = \frac{\sinh\left(\frac{d(o,x)}{2}\right)}{\cosh\left(\frac{d(o,x)}{2}\right)} = \tanh\left(\frac{d(o, x)}{2}\right) < 1.$$

□

Remark 23. *In a complete δ-hyperbolic space (X, d), this implies that given $x \in \partial X$ (the set from the Cauchy completion), then $x \in \partial_\infty X$ (the Gromov boundary). This can also be seen as follows: Scale the metric d such that (X, d) is δ-hyperbolic with $\delta < \ln(2)$. Because a scaling acts on the Gromov boundary by homeomorphisms, this does not change ∂X or ∂_∞ as sets. Then for a Cauchy sequence $x \in \partial X$ we have*

$$\bar{\rho}(x_i, x_j) \leq \rho(x_i, x_j) \leq \lambda \bar{\rho}(x_i, x_j),$$

for some $\lambda > 1$. This implies that $\rho(x_i, x_j) \to 0$, and this in turn implies $(x_i | x_j)_o \to \infty$, therefore $x \in \partial_\infty X$. In particular with the remark from above we have that $\partial X = \partial_\infty X$ as sets.

A

APPENDIX

Books are the treasured wealth of the world and the fit inheritance of generations and nations.
— Henry David Thoreau

A.1 INVARIANCE OF DOUBLING PROPERTY FOR QUASI-METRIC SPACES

The following follows mostly the proof in the main part of the thesis, but in a more general form. It has been moved to the appendix as it might be of limited interest to most readers.

Proposition 25. *Let (X, d) be a K-quasi-metric space [BS07]. Let X_∞ denote the infinite remote set and let $\infty \in X_\infty$, i.e. the space satisfies the relations*

1. $d(x,y) = 0 \iff x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,y) \leq K \max\{d(x,z), d(z,y)\}$ *for all* $x, y, z \in X$ *for which all distances are defined,*
4. $d(x,y) < \infty \iff x, y \in X \setminus X_\infty$.

Let $\lambda : X \to [0,\infty]$, $L > 0$ and $K' \geq K$ be such that $X_\infty = \lambda^{-1}(\infty)$ and

1. $d(x,y) \leq K' \max\{L\lambda(x), L\lambda(y)\}$,
2. $L\lambda(x) \leq K' \max\{d(x,y), L\lambda(y)\}$.

Denote by $X'_\infty := \{\lambda^{-1}(0)\}$. Define a new metric $d_\lambda : (X \times X) \setminus (X'_\infty \times X'_\infty) \to [0,\infty]$ by

1. $d_\lambda(x,y) := \frac{d(x,y)}{\lambda(x)\lambda(y)}$ *for* $x, y \in X \setminus X'_\infty$,

2. $d_\lambda(x,\infty) := d_\lambda(\infty,x) := \frac{L}{\lambda(x)}$ *for* $\infty \in X_\infty$,

3. $d_\lambda(\infty,\infty) = 0$ *for* $\infty \in X_\infty$,

4. $d_\lambda(x,p) := d_\lambda(p,x) := \infty$ *for* $p \in X'_\infty$.

If (X,d) *is doubling with constant* D *then* (X,d_λ) *is doubling with constant at most* $D^{\lceil \log_2(8K'^{10}K)\rceil} + 1$.

Proof. By Prop 5.3.6 in [BS07], d_λ is a K'^2-quasi-metric. In particular we have for all $x,y,z \in X$ for which all distances are defined, that:

$$d_\lambda(x,y) \le K'^2 \max\{d(d,z), d(z,y)\}.$$

Let $x_0 \in X$, $x_0 \neq p \in X'_\infty$ and $r > 0$ and let $B' := B'_r(x_0) := \{x \in X \,|\, d_\lambda(x_0,x) \le r\}$. Consider the following cases

1. If $B' \cap B'_{\frac{1}{2}r}(\infty) \neq \emptyset$, then let $A' := B' \setminus B'_{\frac{1}{2}r}(\infty)$. For all $x,y \in B'$ we have
$$d_\lambda(x,y) = \frac{d(x,y)}{\lambda(x)\lambda(y)} \le K'^2 r,$$
from which it follows that
$$d(x,y) \le K'^2 r\lambda(x)\lambda(y).$$
Furthermore we have for all $x \in A'$ that $d_\lambda(\infty,x) = \frac{L}{\lambda(x)} > \frac{1}{2}r$ and therefore also $\lambda(x) < \frac{2L}{r}$. Combining both equations we get that for all $x,y \in A'$ we have
$$d(x,y) \le K'^2 r \frac{2L}{r}\frac{2L}{r} = \frac{K'^2 4L^2}{r}.$$
Without loss of generality assume $x_0 \in A'$. By the doubling property of (X,d) we can cover $B_{\frac{K'^2 4L^2}{r}}(x_0)$ by at most D^N balls b_i of radius $\frac{K'^2 4L^2}{r} 2^{-N}$. Let $\tilde{b}_i := b_i \cap A'$ then we have for all $x,y \in \tilde{b}_i$:
$$d_\lambda(x,y) \le \frac{\frac{K'^2 4L^2}{r 2^N} K}{\lambda(x)\lambda(y)}.$$

By the assumption there is a $\bar{x} \in B' \cap B'_{\frac{1}{2}r}(\infty)$ and we have for $x \in B'$ that $d_\lambda(x, \bar{x}) \le K'^2 r$, therefore we also have $\frac{L}{\lambda(x)} = d_\lambda(x, \infty) \le K'^4 r$ and $\lambda(x) \ge \frac{L}{K'^4 r}$. In conclusion we get for all $x, y \in \tilde{b}_i$:

$$d_\lambda(x,y) \le \frac{\frac{K'^2 4L^2}{r2^N} K}{\lambda(x)\lambda(y)} \le \frac{\frac{K'^2 4L^2}{r2^N} K}{\frac{L}{K'^4 r}\frac{L}{K'^4 r}} = \frac{K'^{10} K 4r}{2^N}.$$

In particular for $N := \lceil \log_2(8K'^{10}K) \rceil$ we get a cover of B' by at most $D^N + 1$ balls of half the radius.

2. If $B' \cap B'_{\frac{1}{2}r}(\infty) = \emptyset$, then we have $d_\lambda(x_0, \infty) > r$ and $d_\lambda(B', \infty) > \frac{1}{2}r$. For all $y \in B'$ we have $d_\lambda(x_0, y) = \frac{d(x_0,y)}{\lambda(x_0)\lambda(y)} \le r$ and therefore also

$$d(x_0, y) \le r\lambda(x_0)\lambda(y) \le \frac{rL^2}{d_\lambda(\infty, x_0) d_\lambda(\infty, y)} = \frac{rL^2}{d_\lambda(B', \infty)^2}.$$

By the doubling property of (X, d) we can find D^N balls b_i of radius $\frac{rL^2}{d_\lambda(B',\infty)^2} 2^{-N}$ covering B'. Let $\tilde{b}_i := b_i \cap B'$, then we have for any $x, y \in \tilde{b}_i$:

$$d_\lambda(x,y) = \frac{d(x,y)}{\lambda(x)\lambda(y)} \le \frac{K\frac{rL^2 2^{-N}}{d_\lambda(B',\infty)^2}}{\lambda(x)\lambda(y)} = \frac{Kr2^{-N} d_\lambda(\infty, x) d_\lambda(\infty, y)}{d_\lambda(B', \infty)^2}.$$

Furthermore for any $x \in B'$ we have

$$d_\lambda(x, \infty) \le K'^2 \max\{d_\lambda(x_0, x), d_\lambda(x_0, \infty)\} \le K'^2 r \le K'^2 2 d_\lambda(B', \infty).$$

We can combine the estimates to get

$$d_\lambda(x,y) \le \frac{Kr2^{-N} K'^4 4 d_\lambda(B', \infty)^2}{d_\lambda(B', \infty)^2} = Kr2^{-N} K'^4 4.$$

In particular for $N := \lceil \log_2(8KK'^4) \rceil$ we have constructed a covering by D^N balls of radius at most $\frac{1}{2}r$.

□

Proposition 26. *Let (X,d) be a K-quasi-metric space [BS07]. Let X_∞ denote the infinite remote set and let $\infty \in X_\infty$, i.e. the space satisfies the relations*

1. $d(x,y) = 0 \iff x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,y) \leq K\max\{d(x,z), d(z,y)\}$ *for all* $x,y,z \in X$ *for which all distances are defined,*
4. $d(x,y) < \infty \iff x,y \in X \setminus X_\infty$.

Let $\lambda : X \to [0,\infty]$, $L > 0$ and $K' \geq K$ be such that $X_\infty = \lambda^{-1}(\infty)$ and

1. $d(x,y) \leq K' \max\{L\lambda(x), L\lambda(y)\}$,
2. $L\lambda(x) \leq K' \max\{d(x,y), L\lambda(y)\}$.

Denote by $X'_\infty := \{\lambda^{-1}(0)\}$. Define a new metric $d_\lambda : (X \times X) \setminus (X'_\infty \times X'_\infty) \to [0,\infty]$ by

1. $d_\lambda(x,y) := \frac{d(x,y)}{\lambda(x)\lambda(y)}$ *for* $x,y \in X \setminus X'_\infty$,
2. $d_\lambda(x,\infty) := d_\lambda(\infty,x) := \frac{L}{\lambda(x)}$ *for* $\infty \in X_\infty$,
3. $d_\lambda(\infty,\infty) = 0$ *for* $\infty \in X_\infty$,
4. $d_\lambda(x,p) := d_\lambda(p,x) := \infty$ *for* $p \in X'_\infty$.

Let $\theta \leq \frac{1}{K^{19}}$. If (X,d_λ) has a θ-chain, then (X,d) has a $\sqrt[3]{\theta K'^4}$-chain.

Proof. Using the same notation as before in Section 3.3 we note that for all $i \in \{0,\dots,n-1\}$ the following relation holds:

$$\frac{l_i}{\frac{K'^2}{L^2} r_i r_{i+1}} \leq \frac{l_i}{\lambda(x_i)\lambda(x_{i+1})} \leq \frac{l\theta}{\lambda(x_0)\lambda(x_n)} \leq \frac{l\theta}{\frac{1}{K'L} r_0 r_n}.$$

We can apply a similar argument as in Lemma 1 to get an index q for which

$$r_0 \leq \sqrt[3]{\theta K'^4} r_q,$$

and such that for all $i \in \{0, \dots, q-1\}$ we have

$$r_0 > \sqrt[3]{\theta K'^4} r_i.$$

Assume again for a contradiction that $(x_q, x_{x-1}, \dots, x_0, p)$ is not a $\sqrt[3]{\theta K'^4}$-chain. Then for some $i \in \{0, \dots, q-1\}$:

$$\frac{\sqrt[3]{\theta K'^4}^2 r_q}{\frac{K'^2}{L^2} r_0 r_q} \leq \frac{\sqrt[3]{\theta K'^4} r_q}{\frac{K'^2}{L^2} r_i r_q} \leq \frac{\sqrt[3]{\theta K'^4} r_q}{\frac{K'^2}{L^2} r_i r_{i+1}} \leq \frac{\sqrt[3]{\theta K'^4} r_q}{\lambda(x_i)\lambda(x_{i+1})} < \frac{l_i}{\lambda(x_i)\lambda(x_{i+1})} \tag{A.1}$$

$$\leq \frac{\theta l}{\lambda(x_0)\lambda(x_n)} \leq \frac{\theta l}{\frac{1}{K'^2 L^2} r_0 r_n} \tag{A.2}$$

From this it follows that

$$r_n < \sqrt[3]{\theta K'^4} K'^4 l \leq K^{-1} l.$$

□

A.2 ADDITIONAL ALTERNATIVE PROOFS FOR CALCULATING EUCLIDEAN DISTANCE FROM MODEL SPACES

This has originally been used for the proof of the triangle inequality (Theorem 14) for ρ_o in CBB(κ) spaces with $\kappa > 0$, but was no longer necessary when an easier proof was found. For the sake of completeness we included it here in the appendix.

In this section we give additional constructive proofs of some lemmas. Those should also make it more clear that the choice of base point is free.

Alternative Proof of Proposition 20.

$$
\begin{aligned}
&\frac{\operatorname{sn}_\kappa(d(x,y)/2)}{\operatorname{cs}_\kappa(d(x,o)/2)\operatorname{cs}_\kappa(d(y,o)/2)} \\
&\quad= \frac{\sinh(\sqrt{-\kappa}d(x,y)/2)}{\cosh(\sqrt{-\kappa}d(x,o)/2)\cosh(\sqrt{-\kappa}d(y,o)/2)\sqrt{-\kappa}} \\
&\quad= \frac{\frac{\sqrt{-\kappa}\|x-y\|}{\sqrt{1+\kappa\|x\|^2}\sqrt{1+\kappa\|y\|^2}}}{\frac{\sqrt{2+\frac{-2\kappa\|x-o\|^2}{(1+\kappa\|x\|^2)(1+\kappa\|o\|^2)}}}{\sqrt{2}}\frac{\sqrt{2+\frac{-2\kappa\|y-o\|^2}{(1+\kappa\|y\|^2)(1+\kappa\|o\|^2)}}}{\sqrt{2}}\sqrt{-\kappa}} \\
&\quad= \frac{\frac{\|x-y\|}{\sqrt{1+\kappa\|x\|^2}\sqrt{1+\kappa\|y\|^2}}}{\frac{\sqrt{\frac{2(1+\kappa\|x\|^2)(1+\kappa\|o\|^2)-2\kappa\|x-o\|^2}{(1+\kappa\|x\|^2)(1+\kappa\|o\|^2)}}}{\sqrt{2}}\frac{\sqrt{\frac{2(1+\kappa\|y\|^2)(1+\kappa\|o\|^2)-2\kappa\|y-o\|^2}{(1+\kappa\|y\|^2)(1+\kappa\|o\|^2)}}}{\sqrt{2}}} \\
&\quad= \frac{\frac{\|x-y\|}{\sqrt{1+\kappa\|x\|^2}\sqrt{1+\kappa\|y\|^2}}}{\sqrt{\frac{(1+\kappa\|x\|^2)(1+\kappa\|o\|^2)-\kappa\|x-o\|^2}{(1+\kappa\|x\|^2)(1+\kappa\|o\|^2)}}\sqrt{\frac{(1+\kappa\|y\|^2)(1+\kappa\|o\|^2)-\kappa\|y-o\|^2}{(1+\kappa\|y\|^2)(1+\kappa\|o\|^2)}}} \\
&= \frac{\|x-y\|}{\sqrt{(1+\kappa\|x\|^2)(1+\kappa\|o\|^2)-\kappa\|x-o\|^2}\sqrt{(1+\kappa\|y\|^2)(1+\kappa\|o\|^2)-\kappa\|y-o\|^2}} \\
&\quad= \frac{\|x-y\|}{\sqrt{(1+\kappa\|x\|^2)(1+\kappa\|0\|^2)-\kappa\|x\|^2}\sqrt{(1+\kappa\|y\|^2)(1+\kappa\|0\|^2)-\kappa\|y\|^2}} \\
&\qquad = \|x-y\|.
\end{aligned}
$$

□

In the following lemma we show that the euclidean distance of the projection from the sphere to the plane can be calculated from the metric. This is the same result as Proposition 20 but gives another constructive proof.

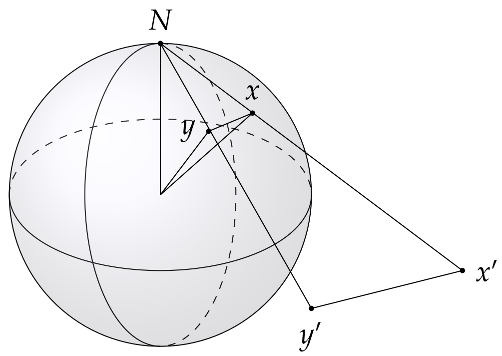

Figure A.1: Projection from the sphere onto the plane

Lemma 18. *Let* $x, y \in X = \partial B_{\frac{1}{\sqrt{\kappa}}}(0) = \{x, |, \|x - 0\| = \frac{1}{\sqrt{\kappa}}\}$ *and denote by* $\angle_z(x, y)$ *the angle between the two lines zx, zy. The spherical metric on X is given by* $d(x, y) = \angle(x, y) = \angle_0(x, y)\frac{1}{\sqrt{\kappa}}$*. Then:*

$$\frac{\text{sn}_\kappa(d(x, y)/2)}{\text{cs}_\kappa(d(x, p)/2)\,\text{cs}_\kappa(d(y, p)/2)} = \|x' - y'\|$$

where x', y' *are the points given by projecting onto the plane orthogonal to* $-pp$ *by some line going through* $-pxx'$ *respectively* $-pyy'$*. See Figure A.1.*

Proof. We can calculate the following distances:

$$\|x - N\| = 2\sin\left(\frac{\angle(N, x)}{2}\right)\frac{1}{\sqrt{\kappa}} = 2\sin(\frac{d(N, x)\sqrt{\kappa}}{2})\frac{1}{\sqrt{\kappa}},$$

$$\|y - N\| = 2\sin(\frac{\angle(N, y)}{2})\frac{1}{\sqrt{\kappa}} = 2\sin(\frac{d(N, y)\sqrt{\kappa}}{2})\frac{1}{\sqrt{\kappa}},$$

$$\|x - y\| = 2\sin(\frac{\angle(x, y)}{2})\frac{1}{\sqrt{\kappa}} = 2\sin(\frac{d(x, y)\sqrt{\kappa}}{2})\frac{1}{\sqrt{\kappa}}.$$

Note that $\angle(N, x) = \pi - \angle(S, x)$, therefore

$$\|x' - N\| = \frac{1}{\cos(\angle_N(x, 0))\sqrt{\kappa}} = \frac{1}{\cos(\pi - \frac{\angle(x,N)}{2} - \frac{\pi}{2})\sqrt{\kappa}} = \frac{1}{\cos(\frac{\angle(x,S)}{2})\sqrt{\kappa}},$$

and

$$\|y'-N\| = \frac{1}{\cos(\angle_N(x,0))\sqrt{\kappa}} = \frac{1}{\cos(\pi - \frac{\angle(y,N)}{2} - \frac{\pi}{2})\sqrt{\kappa}} = \frac{1}{\cos(\frac{\angle(y,S)}{2})\sqrt{\kappa}}.$$

Furthermore because $\sin(\frac{\pi}{2} - x) = \cos(x)$ we get:

$$\|x-N\| = 2\sin(\frac{\angle(N,x)}{2})\frac{1}{\sqrt{\kappa}} = 2\sin(\frac{\pi - \angle(S,x)}{2})\frac{1}{\sqrt{\kappa}} = 2\cos(\frac{\angle(S,x)}{2})\frac{1}{\sqrt{\kappa}},$$

$$\|y-N\| = 2\sin(\frac{\angle(N,y)}{2})\frac{1}{\sqrt{\kappa}} = 2\sin(\frac{\pi - \angle(S,y)}{2})\frac{1}{\sqrt{\kappa}} = 2\cos(\frac{\angle(S,y)}{2})\frac{1}{\sqrt{\kappa}}.$$

Applying the law of cosines to the above distances we can calculate the angle between the lines Nx, Ny:

$$\alpha := \angle_N(x,y) = \arccos(\frac{\|x-N\|^2 + \|y-N\|^2 - \|x-y\|^2}{2\|x-N\|\|y-N\|}).$$

Then applying the law of cosines again we can calculate:

$$
\begin{aligned}
\|x' - y'\|^2 &= \|N - x'\|^2 + \|N - y'\|^2 - 2\|N - x'\|\|N - y'\| \cos(\alpha) \\
&= \frac{1}{\cos(\frac{\angle(x,S)}{2})^2 \kappa} + \frac{1}{\cos(\frac{\angle(y,S)}{2})^2 \kappa} - 2\frac{1}{\cos(\frac{\angle(x,S)}{2})\sqrt{\kappa}} \frac{1}{\cos(\frac{\angle(y,S)}{2})\sqrt{\kappa}} \cos(\alpha) \\
&= \frac{\cos(\frac{\angle(y,S)}{2})^2 + \cos(\frac{\angle(x,S)}{2})^2 - 2\cos(\frac{\angle(x,S)}{2})\cos(\frac{\angle(y,S)}{2})\cos(\alpha)}{\cos(\frac{\angle(x,S)}{2})^2 \cos(\frac{\angle(y,S)}{2})^2 \kappa} \\
&= \frac{\cos(\frac{\angle(y,S)}{2})^2 + \cos(\frac{\angle(x,S)}{2})^2}{\cos(\frac{\angle(x,S)}{2})^2 \cos(\frac{\angle(y,S)}{2})^2 \kappa} \\
&\quad - \frac{2\cos(\frac{\angle(x,S)}{2})\cos(\frac{\angle(y,S)}{2}) \frac{4\cos(\frac{\angle(S,x)}{2})^2 \frac{1}{\kappa} + 4\cos(\frac{\angle(S,y)}{2})^2 \frac{1}{\kappa} - 4\sin(\frac{\angle(x,y)}{2})^2 \frac{1}{\kappa}}{8\cos(\frac{\angle(S,x)}{2})\cos(\frac{\angle(S,y)}{2})\frac{1}{\kappa}}}{\cos(\frac{\angle(x,S)}{2})^2 \cos(\frac{\angle(y,S)}{2})^2 \kappa} \\
&= \frac{\cos(\frac{\angle(y,S)}{2})^2 + \cos(\frac{\angle(x,S)}{2})^2}{\cos(\frac{\angle(x,S)}{2})^2 \cos(\frac{\angle(y,S)}{2})^2 \kappa} \\
&\quad - \frac{\cos(\frac{\angle(x,S)}{2})\cos(\frac{\angle(y,S)}{2}) \frac{\cos(\frac{\angle(S,x)}{2})^2 \frac{1}{\kappa} + \cos(\frac{\angle(S,y)}{2})^2 \frac{1}{\kappa} - \sin(\frac{\angle(x,y)}{2})^2 \frac{1}{\kappa}}{\cos(\frac{\angle(S,x)}{2})\cos(\frac{\angle(S,y)}{2})\frac{1}{\kappa}}}{\cos(\frac{\angle(x,S)}{2})^2 \cos(\frac{\angle(y,S)}{2})^2 \kappa} \\
&= \frac{\cos(\frac{\angle(y,S)}{2})^2 + \cos(\frac{\angle(x,S)}{2})^2 - \cos(\frac{\angle(S,x)}{2})^2 - \cos(\frac{\angle(S,y)}{2})^2 + \sin(\frac{\angle(x,y)}{2})^2}{\cos(\frac{\angle(x,S)}{2})^2 \cos(\frac{\angle(y,S)}{2})^2 \kappa} \\
&= \frac{\sin(\frac{\angle(x,y)}{2})^2}{\cos(\frac{\angle(x,S)}{2})^2 \cos(\frac{\angle(y,S)}{2})^2 \kappa} \\
&= \frac{\sin(\frac{d(x,y)\sqrt{\kappa}}{2})^2}{\cos(\frac{d(x,S)\sqrt{\kappa}}{2})^2 \cos(\frac{d(y,S)\sqrt{\kappa}}{2})^2 \kappa} \\
&= \frac{\mathrm{sn}_\kappa(\frac{d(x,y)}{2})^2}{\mathrm{cs}_\kappa(\frac{d(x,S)}{2})^2 \mathrm{cs}_\kappa(\frac{d(y,S)}{2})^2}.
\end{aligned}
$$

□

A.3 A DISCRETE GRAPH AS A MODEL OF THE HYPERBOLIC PLANE

The following example has been constructed together with Viktor Schroeder, who helped complete the proof which shows that the graph it is 1-hyperbolic.

OUTLINE OF CONSTRUCTION. We construct a metric graph V^+ and show that this graph must be 1-hyperbolic. We then scale the metric by δ and show that for $\delta > \ln(2)$ we can no longer get bi-Lipschitz equivalence in Theorem 16.

A.3.1 *Construction of Graph*

Consider the planar graph G embedded in $\mathbb{R}^2$ given by:

vertices: The vertices are given by the set $V = \{(k \cdot 2^{-n}, n) \in \mathbb{R}^2 \,|\, k, n \in \mathbb{Z}\}$,

edges: There are horizontal edges (joining two vertices of the same level)

$$E_{\text{hor}} = \{\{k \cdot 2^{-n}, n), ((k+1)2^{-n}, n)\} \,|\, k, n \in \mathbb{Z}\}$$

and vertical edges (joining two vertices of neighboring levels):

$$\begin{aligned} E_{\text{vert}} = \Big\{ &\{(k \cdot 2^{-n}, n), ((2k-1)2^{-n-1}, n+1)\}, \\ &\{(k \cdot 2^{-n}, n), ((2k)2^{-n-1}, n+1)\}, \\ &\{(k \cdot 2^{-n}, n), ((2k+1)2^{-n-1}, n+1)\} \,|\, k, n \in \mathbb{Z} \Big\} \end{aligned}$$

Thus every vertex v is in two horizontal edges joining it with "right" and "left" neighbors of the same level. It is contained in three edges joining it to neighbors of one level higher. It is contained in one or two edges joining it to neighboring vertices of one level lower: $v = (k, 2^{-n}, n)$ has two neighbors of level $(n-1)$ if k is odd and one if k is even.

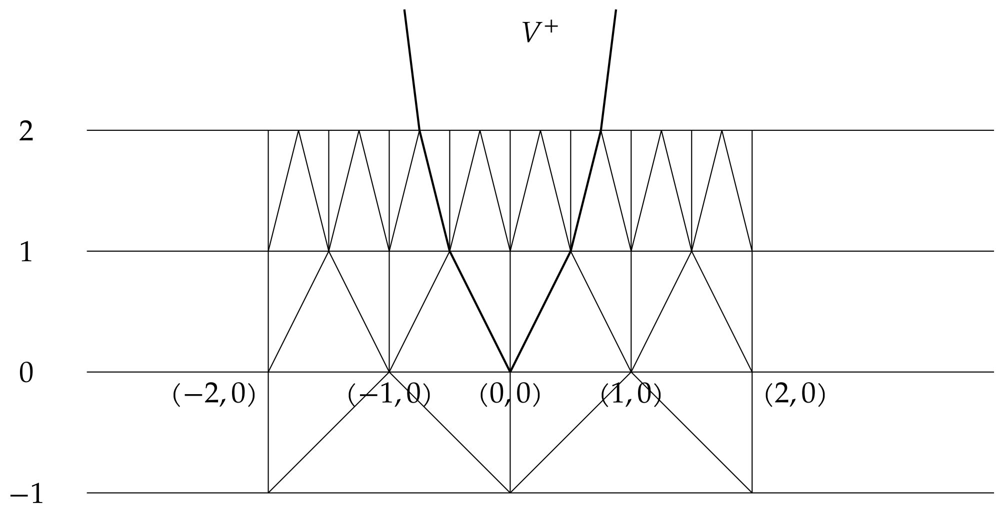

Figure A.2: An example of a discrete hyperbolic space

We consider on V the path metric d of the graph, where we give every edge the length 1 (if scaled to length $\ln(2)$ we obtain a discrete "model" of $\mathbb{H}^2$).

We define

$$V^+ = \{v = (k2^{-n}, n) \mid n \geq 0, |k2^{-n}| < 1, n \in \mathbb{Z}\}.$$

Note that for $x = (-\frac{1}{2}, 2), y = (\frac{1}{2}, 2), z = (0, 2), o = (0, 0)$ we have

$$d(x, z) = 2 = d(z, y), \quad d(x, y) = 4$$

and

$$d(o, x) = d(o, y) = d(o, z) = 2$$

therefore the Gromov products satisfy:

$$(x|z)_o = (y|z)_o = 1, \quad (x|y)_o = 0.$$

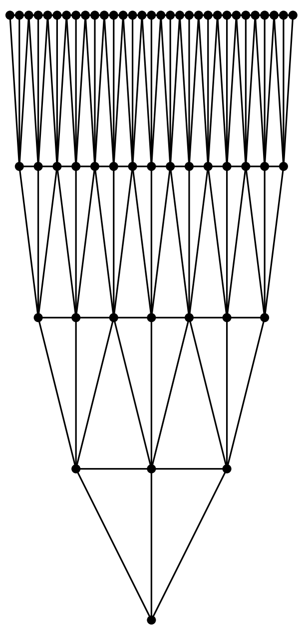

Figure A.3: The discrete hyperbolic space V^+

Note that a space is Gromov δ-hyperbolic if for all x, y, z, o:

$$(x|z)_o \geq \min((x|y)_o, (y|z)_o) - \delta.$$

Therefore in our space we must have that $\delta \geq 1$.

A.3.2 Geodesics in G

Definition 35. *A sequence* $v_o, v_1, \dots, v_k$ *of vertices is a* geodesic *if* $d(v_0, v_k) = k$*. This implies that* $d(v_i, v_j) = |j - i| \quad \forall i, j \in \{0, \dots, k\}$*. A geodesic* $v_0, \dots, v_k$ *is called* vertical*, if* $l(v_o) - l(v_k)| = k$*. This implies that either:*

$$l(v_j) = l(v_0) + j \quad \text{for all } j \text{ or,}$$
$$l(v_j) = l(v_0) - j \quad \text{for all } j.$$

A geodesic $v_0, \dots, v_k$ *is* horizontal *if* $l(v_0) = \dots = l(v_k)$*. By inspection on the graph it is easy to see that a horizontal geodesic has length* $k \leq 4$*.*

Definition 36. *Given* $x, y, z \in V$ *we say:* z *lies* between x *and* y *if* $d(x, z) + d(z, y) = d(x, y)$*. This implies that there exists a geodesic from* x *to* y *going through* z*.*

Definition 37. *Let* $x, y \in V$ *then*

$$\text{minlev}(x, y) = l$$

is the minimum level, such that there exists z *between* x *and* y *with* $l(z) = \text{minlev}(x, y)$*.*

Lemma 19. *Let* $x, y \in V$ *with* $d(x, y) = k$ *then there exist* $0 \leq r \leq s \leq k$ *and a geodesic* $x = v_0, \dots, v_k = y$ *such that* $v_0, \dots, v_r$ *is vertical with decreasing level,* $v_r, \dots, v_s$ *is horizontal with length* $s - r \leq 2$ *and* $v_s, \dots, v_k$ *is vertical with increasing level. And furthermore* $l(v_r) = \dots = l(v_s) = \text{minlev}(x, y)$*.*

Proof. (a) First note that there is no geodesic w_i, w_{i+1}, w_{i+2} with $l(w_i) < l(w_{i+1}) > l(w_{i+2})$. Now we have the following replacement procedure:

(b1) Let w_i, w_{i+1}, w_{i+2} be a geodesic with $l(w_i) = l(w_{i+1}) > l(w_{i+2})$. This can be replaced by a geodesic w_i, w'_{i+1}, w_{i+2} with $l(w_i) > l(w'_{i+1}) = l(w_{i+2})$.

(b2) Let w_i, w_{i+1}, w_{i+2} with $l(w_i) < l(w_{i+1}) = l(w_{i+2})$. This can be replaced by w_i, w'_{i+1}, w_{i+2} with $l(w_i) = l(w'_{i+1}) < l(w_{i+2})$.

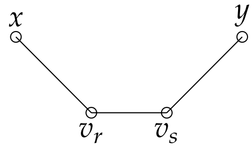

Figure A.4: A standard geodesic in the discrete hyperbolic space

(b3) Let $w_i, w_{i+1}, w_{i+2}, w_{i+3}$ with $l(w_i) = \dots = l(w_{i+2})$. This can be replaced by $w_i, w'_{i+1}, w'_{i+2}, w_{i+3}$ with $l(w_i) > l(w'_{i+1}) = l(w'_{i+2}) < l(w_{i+3})$.

Points (a), (b1), (b2), (b3) can be easily seen by inspection. Now choose $z \in G$ with $l(z) = \text{minlev}(x, y)$ and choose a geodesic from x to y which contains z. Using the procedures (b1), (b2), (b3) one can deform this geodesic. □

Definition 38. *We call a geodesic standard if it satisfies the properties of the lemma.*

Remark 24. *It is not difficult to show, that the points $v_r, \dots, v_s$ are exactly the points z with $l(z) = \text{minlev}(x, y)$ such that z is between x and y.*

Definition 39. *$a(x, y) := (s - r)$, i.e. the number of horizontal edges in the standard geodesic.*

We have $0 \leq a(x, y) \leq 2$.

Definition 40. *We write $[x, v_r, v_s, y]$ for a standard geodesic. See Figure A.4*

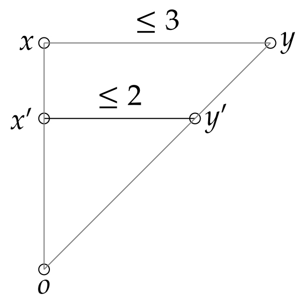

Figure A.5: Points on lower levels move closer together

A.3.3 *Properties of V^+*

We now study properties of $V^+ \subset V$.

OBSERVATION: V^+ is a totally geodesic subset: i.e. every geodesic with endpoints in V^+ is completely contained in V^+.

V^+ consists of the vertices $x \in V$ with $l(x) \geq 0$ such that there exists a vertical geodesic from o to x.

Lemma 20. *Let $x \in V^+$ and $l(x) = h$. Then there exists a geodesic $o = v_0, \dots, v_h = x$, with $l(v_i) = i$.*

Definition 41. *If $x, y \in V^+$, $l(x) = l(y) = m$, then we denote by $d^m(x, y)$ the distance on the level, i.e. the length of the minimal horizontal path joining x and y.*

Lemma 21. *Let $x, y \in V^+$ with $l(x) = l(y) = k$.*

1. *If $d(x, y) \leq 1$ and x' resp. y' are between o and x resp. y and $l(x') = l(y')$, then $d(x', y') \leq 1$.*

2. *If $d^k(x, y) \leq 3$ and x' is between o and x with $l(x') = m < k$. Then there exists y' between o and y with $l(y') = m$ and $d^m(x', y') \leq 2$.*

Proof. By inspection of the graph. □

We now show that:

Theorem 20. *$x, y, z \in V^+$, then* $(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - 1$.

Proof. We will use the following formula for the $(x|y)_o$, $x, y \in V^+$:

$$(x|y)_o = \text{minlev}(x, y) - \frac{a(x, y)}{2}.$$

Indeed $d(x, y) = l(x) + l(y) - 2\,\text{minlev}(x, y) + a(x, y)$ thus $(x|y)_o = \frac{1}{2}(l(x) + l(y) - d(x, y))$ gives the formula. Note that $d(x, o) = l(x)$ by Lemma 20. We can now assume that $(x|y)_o$ is the minimum among the three Gromov-products. We can assume without loss of generality that $m = \text{minlev}(x, z) \leq \text{minlev}(z, y) = k$. Let $[x, p_1, p_2, z]$ and $[z, p_3, p_4, y]$ be standard geodesics.

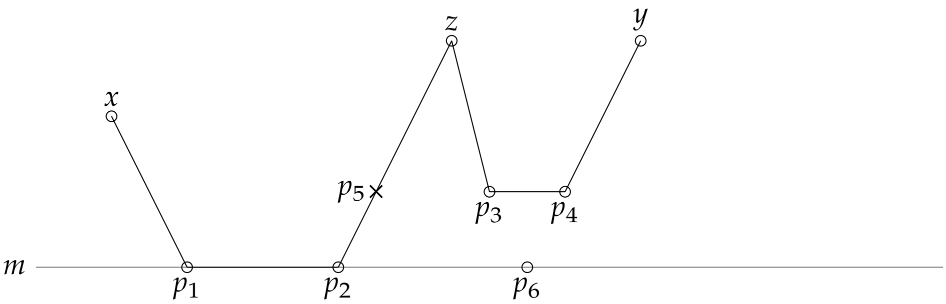

Figure A.6: Proof of Theorem 20, case $k > m$

case $k > m$: Choose p_5 between p_2 and y with $l(p_5) = k$. Since p_5 and p_3 are both between 0 and y we have $d^k(p_5, p_3) \leq 1$ by Lemma 21 (1). This implies $d^k(p_5, p_4) \leq 3$. Thus there exists p_6 between p_4 and 0 on level m with $d^m(p_2, p_6) \leq 2$. Thus $d(x, y) \leq d(x, p_1) + a(x, z) + d(p_2, p_6) + d(p_6, z) \leq l(x) + l(y) - 2m + a(x, z) + 2$. Thus $(x|y)_o = \frac{1}{2}(l(x) + l(y) - d(x, y) \geq \frac{1}{2}(2m - a(x, z) + 2) = (x|z)_o - 1$.

case $k = m$: By Lemma 21 (1) we have $d^m(p_2, p_3) \leq 1$.

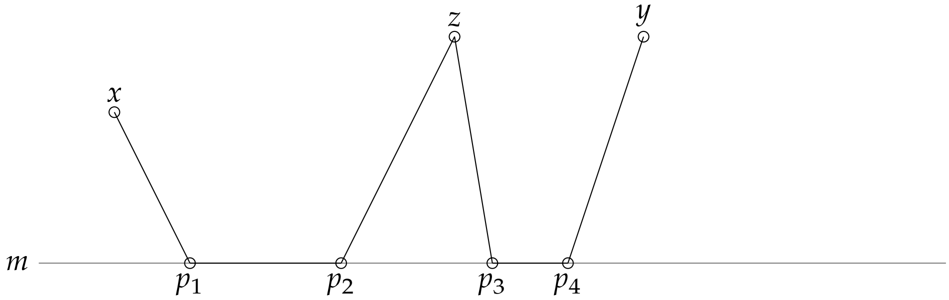

Figure A.7: Proof of Theorem 20, case $k = m$

subcase $\max(a(x,z), a(y,z)) = 2$**:** In this situation

$$\min((x|z)_o, (z|y)_o) = (m-1).$$

Now $d^m(p_1, p_4) \leq 5$, this implies (by inspection of the graph) that $d(p_1, p_4) \leq 4$, and hence

$$d(x,y) \leq l(x) + l(y) - 2m + 4,$$

and furthermore $(x|y)_o \geq m - 2$.

subcase $\max(a(x,z), a(y,z)) = 1$**:** Then

$$\min((x|z)_o, (z|y)_o) = m - \frac{1}{2}.$$

Now $d^m(p_1, p_4) \leq 3$ and hence

$$d(x,y) \leq l(x) + l(y) - 2m + 3.$$

I.e., $(x|y)_o \geq m - \frac{3}{2}$.

subcase $\max(a(x,z), a(y,z)) = 0$**:** $a(x,z) = a(y,z) = 0$, then

$$\min((x|z)_o, (z|y)_o) = m$$

and $d(x,y) \leq l(x)+l(y)-2m+1$ which implies that $(x|y)_o \geq m - \frac{1}{2}$.

□

A.3.4 *ln*(2) *is Optimal in Theorem 16*

Corollary 3. *Let* (V^+, d) *be the space we just defined, where we scale the metric by the factor* δ. *Let*

$$\rho(x,y) = \frac{\sinh(d(x,y)/2)}{\cosh(d(x,o)/2)\cosh(d(y,o)/2)}$$

where $o = (0,0)$ *and let* $\overline{\rho}$ *be the metric after applying the Frink construction. Then* ρ *is not bi-Lipschitz to* $\overline{\rho}$ *for* $\delta > \ln(2)$.

Proof. Let $x_n = (1 - 2^{-n}, n)$ and $y_n = (-1 + 2^{-n}, n)$, then

$$\begin{aligned}
\rho(x_n, y_n) &= \frac{\sinh(d(x_n,y_n)/2)}{\cosh(d(x_n,o)/2)\cosh(d(y_n,o)/2)} \\
&= \frac{\sinh(d(x_n,o)/2 + d(o,y_n)/2)}{\cosh(d(x_n,o)/2)\cosh(d(y_n,o)/2)} \\
&= \frac{\sinh(n\delta)}{\cosh(n\delta/2)\cosh(n\delta/2)} \\
&\overset{n\to\infty}{\longrightarrow} 1.
\end{aligned}$$

On the other hand for $\overline{\rho}(x_n, y_n)$ we get

$$\begin{aligned}
\overline{\rho}(x_n, y_n) &\le \sum_{i=0}^{2^{n+1}-2} \frac{\sinh(\delta/2)}{\cosh(d(x,o)/2)\cosh(d(y,o)/2)} \\
&\le \sum_{i=0}^{2^{n+1}-2} \frac{\sinh(\delta/2)}{\cosh(n\delta/2)\cosh(n\delta/2)} \\
&\le (2^{n+1} - 2)\frac{\sinh(\delta/2)}{\cosh(n\delta/2)\cosh(n\delta/2)} \\
&\overset{n\to\infty}{\longrightarrow} (2^{n+1} - 2)\exp(\delta/2 - n\delta).
\end{aligned}$$

This converges to 0 iff $\delta > \ln(2)$ and diverges otherwise. □

A.3.5 *The Constants in Lemma 8 are Optimal*

Corollary 4. *Let $\delta > 0$. There exists a δ-hyperbolic space (X,d) and for any large $s_0 > 0$, there exist points $x, y, z \in X$ such that*

$$d(x,o), d(y,o), d(z,o), d(x,y), d(y,z), d(x,z) \geq s_0,$$

and $(\rho(x,y), \rho(y,z), \rho(z,x))$ can satisfy at most the $\exp(\delta)$-quasi-metric inequality.

Proof. Let (X,d) be the graph space defined above and let $o = (0,0)$, $x_n = (-1/2, n)$, $y_n = (0,n)$ and $z_n = (1/2, n)$. Then

$$\begin{aligned} \rho(x_n, y_n) &= \rho(y_n, z_n) \\ &= \frac{\sinh(d(x_n,y_n)/2)}{\cosh(d(x_n,o)/2)\cosh(d(y_n,o)/2)} \\ &= \frac{\sinh(\frac{(2^n-2)\delta}{2})}{\cosh(\frac{2\delta}{2})\cosh(\frac{2\delta}{2})} \end{aligned}$$

and

$$\begin{aligned} \rho(x_n, z_n) &= \frac{\sinh(d(x_n,y_n)/2)}{\cosh(d(x_n,o)/2)\cosh(d(y_n,o)/2)} \\ &= \frac{\sinh(\frac{2^n\delta}{2})}{\cosh(\frac{2\delta}{2})\cosh(\frac{2\delta}{2})}. \end{aligned}$$

In particular

$$\lim_{n\to\infty} \frac{\rho(x_n, z_n)}{\rho(x_n, y_n)} = \exp(\delta).$$

□

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